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Abstract. *Kernel density estimators technique has been successfully applied to efficient Bit Error Rate (BER) computation issue under a diversity of simulation frameworks. However, as contemporary and emerging digital communication systems are increasingly provided with advanced transceivers, it is questionable if the Symbol Error Rate (SER) can be anyway derived from the BER. This paper investigates for a direct way to efficiently compute the SER. Focusing on the ubiquitous multi-level Quadrature Amplitude Modulation (QAM) transmission schemes, a Gaussian kernel-based estimator is designed. Simulation of the* 4*-QAM transmission scheme under various channel models shows that the proposed estimator can achieve efficient estimations with a very high degree of accuracy and reliability.*

Key words: kernel density estimators, probability density function estimation, bit error probability estimation, symbol error probability estimation.

1. Introduction

The issue of efficient error probability estimation for the performance evaluation of digital communication systems has been an active subject of research during the 1970s. Attempts to estabblish analytical methods of Bit Error Probability (BEP) estimation have been noted [1], [2] but did not meet success. However, simulation-based techniques rapidly showed to be more successfull. In [3], a complete review dealt in details with simulation-based techniques namely the standard Monte Carlo (MC) simulation and variance-reducing techniques such as importance sampling (also called modified Monte Carlo) [4], extreme-value theory [5], tail extrapolation [6] and quasi-analytical estimation [3].

The standard MC technique is the most general of all existing simulation-based techniques of error probability estimation. Indeed, this technique does not need to be adapted to the communication system specificities. It is an intuitive and easy way to compute an estimate of the error probability. It makes a comparison between the transmitted and the received data sequences and then proceeds with the errors counting. The error rate, an estimate of the error probability, is then derived as the ratio of the number of observed errors to the total size of the transmitted sequence. So, the standard MC appears to be a universal technique for the computation of Bit Error Rate (BER) or Symbol Error Rate (SER). Unfortunately, it is revealed to be the most costly computation of error rate simulation methods [3]. To achieve a given level of accuracy and reliability, the standard MC can require large sample size causing the computational cost to be increasingly high whilst the error rate to compute gets smaller.

To overcome the standard MC method computational cost, the family of variance-reducing techniques and more specifically the importance sampling technique has been well investigated. Very high efficiency of factor theoretically going up to hundreds were reported regarding the importance sampling technique. However, this most successfull variance-reducing technique requires a prior knowledge about the underlying probability density function (pdf) of the observations based on which the estimation is made. A more recent approach for efficient error probability estimation consists of using kernel density estimators [7], [8].

The approach based on kernel density estimators allows the error rate to be computed without requiring any prior knowledge about the underlying pdf of the observations. The pdf of soft observations, i.e., those observations sampled from the channel output and serving for hard decision making about the transmitted symbols, is estimated in a non-parametric manner using a kernel density estimator. Then, based on a selected kernel density estimator, some mathematical manipulations allow an equation of the error probability estimate to be derived. Finally, using soft observations at the receiver-end, the estimate of the error probability is computed by simulation.

The first works related to kernel-based error probability estimation dealt with the BER computation. They include works reported in [9], [10] and to some extent, that work [11] based on Parzen's pdf estimator. Parzen's pdf estimator is a more general class that includes kernel estimators. The latest kernel-based BEP estimation works were reported in [12] and [13]. Otherwise, investigations dealing with Symbol Error Probability (SEP) estimation led to results reported in [14] and [15]. Herein, this paper aims at extending the work presented in [15]. The BEP and the SEP are two parameters commonly used in digital communications domain for systems performance evaluation. For Gray coded mappings over the usual channels and for large ranges of signal to noise ratios, it is mathematically demonstrated that the SEP estimate (i.e., the SER) can be deduced from the BEP estimate (i.e., the BER) and inversly [16]. However, in the contemporary and emerging digital communication systems characterised by advanced and complex transceivers, it is questionable if the SER can be anyway deduced from the BER. Far from answering this question, the purpose of this paper is to offer a direct and efficient approach to compute the SER in any digital communication system where the Quadrature Amplitude Modulation (QAM) schemes take place.

In the remainder of this paper, we first formalise in Section 2 the SEP as a function of the pdf of the received soft observations. We then present in Section 3 the SEP estimate based on the kernel density estimators approach. From the general design of the M-QAM SEP estimate made in Section 3, we define in Section 4 the special case of 4-QAM SEP estimator. We specify in Section 5 the simulation frameworks, report and analyse the simulation results. Finally, we conclude the paper in Section 6.

2. Formal Symbol Error Probability

To formulate the SEP estimation problem, let us consider a digital communication system with rectangular M-QAM (4-QAM, 16-QAM, 64-QAM, etc.) constellations. Let $(s_i)_{1\leq i\leq N}$ be a set of N independent and identically distributed information symbols. Each s_i takes its value into the alphabet of waveforms $\{S_1, S_2, \ldots, S_M\}$. At the receiver-end, we assume the presence of soft symbols which are noisy copies of the transmitted symbols s_i. Let us denote by Z, the multivariate Random Variable (RV) describing the received soft observations, such as $Z = (X, Y)$ where X and Y are real RVs. Let $(Z_i)_{1\leq i\leq N}$ be the sequence of the realisations of Z where each $Z_i = (X_i, Y_i)$ corresponds to the soft decision statistic that serves for the computation of the hard decision about the transmitted information symbol s_i. Let us assume a clustering of the received soft observations Z_i into M clusters $(\mathcal{C}_m)_{1\leq m\leq M}$; each with cardinality N_m. Then, the *a priori* probability that the transmitted symbol s_i is equal to S_m is

$$\pi_m \triangleq Pr[s_i = S_m], \tag{2.1}$$

where $m \in \{1, 2, \ldots, M\}$, $Pr[.]$ denotes the probability of a given event and $\pi_1+\pi_2+\ldots+\pi_M = 1$. The soft observations $(Z_i)_{1\leq i\leq N}$ are RVs having the same pdf $f_{Z,N}(x, y)$. The pdf $f_{Z,N}(x, y)$ is a mixture of M marginal multivariate conditional pdfs and is given by

$$f_{Z,N}(x, y) = \sum_{m=1}^{M} \pi_m f_{Z,N_m}^{S_m}(x, y), \tag{2.2}$$

where $f_{Z,N_m}^{S_m}(x, y)$ is the conditional multivariate pdf of Z_i conditional to $s_i = S_m$. The SEP is therefore given by:

$$p_{e,N} = \sum_{m=1}^{M} \pi_m p_{e,N_m}, \tag{2.3}$$

where,

$$p_{e,N_m} = 1 - \iint_{\mathcal{D}_m} f_{Z,N_m}^{S_m}(x, y)\,\mathrm{d}x\,\mathrm{d}y, \tag{2.4}$$

where $\mathcal{D}_m = \{Z \mid m = \underset{1\leq k\leq M}{\operatorname{argmin}}\, d(Z, S_k)\}$, with $d(.,.)$ denoting the euclidian distance, is the decision-region associated to S_m.

3. Gaussian Kernel-Based M-QAM SEP Estimation

In this Section, we proceed with a general formulation, based on the kernel density estimators, of the M-QAM SEP estimation problem. We establish a general expression of the Gaussian kernel-based SEP estimate for any M-QAM transmission scheme.

As shown in equations (2.3) and (2.4), the analytical expressions of the marginal conditional pdfs $f_{Z,N_1}^{S_1}(x, y), f_{Z,N_2}^{S_2}(x, y), \ldots, f_{Z,N_M}^{S_M}(x, y)$ are required to compute the SEP $p_{e,N}$. Unfortunately, these conditional pdfs depend on the communication channel model and receiver scheme so that their exact parametric model is not always available. However, it is possible to resort to a non-parametric estimation technique such as the kernel density estimators. Therefore, let us denote by $\hat{f}_{Z,N_m}^{S_m}(x, y)$ the kernel-based estimate of the marginal multivariate pdf $f_{Z,N_m}^{S_m}(x, y)$. Then, the kernel method allows $\hat{f}_{Z,N_m}^{S_m}(x, y)$ to be written as follows,

$$\hat{f}_{Z,N_m}^{S_m}(x,y) = \frac{1}{N_m}\sum_{Z_i\in\mathcal{C}_m}\frac{1}{h^x_{N_m}}K\left(\frac{x-X_i}{h^x_{N_m}}\right)\times\frac{1}{h^y_{N_m}}K\left(\frac{y-Y_i}{h^y_{N_m}}\right), \tag{3.1}$$

where $h^x_{N_m}$ (resp. $h^y_{N_m}$) is the smoothing parameter depending on the soft values $X_1, X_2, \ldots, X_{N_m}$ (resp. $Y_1, Y_2, \ldots, Y_{N_m}$) that are associated with those soft observations $Z_1, Z_2, \ldots, Z_{N_m}$ which are classified into the cluster $\mathcal{C}_m$. The clusters $(C_m)_{1\leq m\leq M}$ are the result of a classification procedure. Each cluster C_m is assumed containing the soft observations $Z_1, Z_2, \ldots, Z_{N_m}$ which likely belong to the decision-region D_m.The function $K(.)$ is any pdf, called the kernel, assumed to be an even and regular function with zero mean and unit variance.

From (2.3) and (2.4) in which we insert the expressions of the marginal conditional pdfs estimates, we can derive the SEP estimate $\hat{p}_{e,N}$ as follows,

$$\hat{p}_{e,N} = \sum_{m=1}^{M}\hat{\pi}_m\hat{p}_{e,N_m}, \tag{3.2}$$

where $\hat{p}_{e,N_m}$, the marginal SEP, is given by

$$\hat{p}_{e,N_m} = 1 - \iint_{\mathcal{D}_m}\hat{f}_{Z,N_m}^{S_m}(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.3}$$

Since rectangular QAM constellations are considered, the decision-regions $(\mathcal{D}_m)_{1\leq m\leq M}$, are such as $\mathcal{D}_m = [a_m, b_m]\times[c_m, d_m]$. Then, taking into account the expression of $\hat{f}_{Z,N_m}^{S_m}(x,y)$ as given in (3.1), we get

$$\hat{p}_{e,N_m} = 1 - \frac{1}{N_m}\sum_{Z_i\in\mathcal{C}_m}\left\{\int_{a_m}^{b_m}\frac{1}{h^x_{N_m}}K\left(\frac{x-X_i}{h^x_{N_m}}\right)\mathrm{d}x\right\}\times\left\{\int_{c_m}^{d_m}\frac{1}{h^y_{N_m}}K\left(\frac{y-Y_i}{h^y_{N_m}}\right)\mathrm{d}y\right\}. \tag{3.4}$$

Using the changes of variables such as $u = (x-X_i)/h^x_{N_m}$, $v = (y-Y_i)/h^y_{N_m}$ and setting $\alpha_m = (a_m - X_i)/h^x_{N_m}$, $\beta_m = (b_m - X_i)/h^x_{N_m}$, $\gamma_m = (c_m - Y_i)/h^y_{N_m}$ and $\xi_m = (d_m - Y_i)/h^y_{N_m}$, we get

$$\hat{p}_{e,N_m} = 1 - \frac{1}{N_m}\sum_{Z_i\in\mathcal{C}_m}\left\{\int_{\alpha_m}^{+\infty}K(u)\,\mathrm{d}u - \int_{\beta_m}^{+\infty}K(u)\,\mathrm{d}u\right\}\times\left\{\int_{\gamma_m}^{+\infty}K(v)\,\mathrm{d}v - \int_{\xi_m}^{+\infty}K(v)\,\mathrm{d}v\right\}. \tag{3.5}$$

From this general expression of $\hat{p}_{e,N_m}$ given by (3.5), if a Gaussian kernel function is selected, then K is a zero mean unit variance Gaussian distribution. A general expression of the Gaussian kernel-based SEP estimate for any M-QAM transmission scheme is therefore expressed as in (3.2) with in addition the following more convenient expression of $\hat{p}_{e,N_m}$:

$$\hat{p}_{e,N_m} = 1 - \frac{1}{N_m}\sum_{Z_i\in\mathcal{C}_m}\left\{Q\left(\frac{a_m-X_i}{h^x_{N_m}}\right) - Q\left(\frac{b_m-X_i}{h^x_{N_m}}\right)\right\}\times\left\{Q\left(\frac{c_m-Y_i}{h^y_{N_m}}\right) - Q\left(\frac{d_m-Y_i}{h^y_{N_m}}\right)\right\}, \tag{3.6}$$

where the function $Q(.)$ is the complementary unit cumulative Gaussian distribution and is such as $Q(u) = \int_u^{+\infty}(1/\sqrt{2\pi})exp(-t^2/2)\,\mathrm{d}t$.

4. Kernel-Based 4-QAM SEP Estimator

A kernel-based error probability estimator is completely defined by the selection of the kernel function K and its smoothing parameter. The selection of the kernel function can be guided by the density function under estimation. For instance, infinite support distributions such as the Gaussian distribution are suitable as kernel functions whenever the observed samples are distributed over a large scale. Finite support distributions also exist. Further information about them can be found in [17]. The selection of the smoothing parameter is generally concerned with more labor. It has been proven in [8] that the pdf estimation accuracy mainly depends on it. It is also noticed [17] that there is no universal smoothing parameter; i.e., every case of study should accomodate with the judicious method of the optimal smoothing parameter selection. A multitude of methods have been developed for this purpose. Some of them are used in [18] and [19]. A complete review covering these methods can be found in [17]. Several kernel-based BEP estimators (e.g., [9] and [10]) have been built based on the Gaussian kernel.

By the general expression of the SEP estimate given by (3.2) and (3.6), we design a Gaussian kernel-based SEP estimator for any M-QAM transmission scheme except that the smoothing parameters $h^x_{N_m}$ and $h^y_{N_m}$ are not selected yet. Thus, to complete the estimator design and proceed herein with its evaluation in the special case of the 4-QAM transmission scheme, we choose the smoothing parameters as defined in [10] and given by:

$$\begin{cases} h^x_{N_m} = \left(\frac{4}{3}\right)^{\frac{1}{5}} N_m^{-1/5}\sigma^x_{N_m} \\ h^y_{N_m} = \left(\frac{4}{3}\right)^{\frac{1}{5}} N_m^{-1/5}\sigma^y_{N_m}, \end{cases} \tag{4.1}$$

where $(\sigma^x_{N_m})^2$ (resp. $(\sigma^y_{N_m})^2$) is the variance of the soft values $X_1, X_2, \ldots, X_{N_m}$ (resp. $Y_1, Y_2, \ldots, Y_{N_m}$). In [15], we reported simulation results of an SEP estimator based on the smoothing parameter of (4.1) and provided illustrations of performance regarding 4-QAM and 16-QAM transmission schemes. Without any loss of generality, we'll focus the performance evaluation of the proposed kernel-based SEP estimator on the 4-QAM transmission scheme. In this particular case, the decision-regions are such as: $\mathcal{D}_1 = [0,+\infty[\times[0,+\infty[$, $\mathcal{D}_2 = [0,+\infty[\times]-\infty,0]$, $\mathcal{D}_3 =]-\infty,0]\times]-\infty,0]$ and $\mathcal{D}_4 =]-\infty,0]\times[0,+\infty[$. Thus, from (3.6), $Q\left((a_m - X_i)/h^x_{N_m}\right) = 1$ and $Q\left((c_m - Y_i)/h^y_{N_m}\right) = 1$ as $a_m \to -\infty$ and $c_m \to -\infty$ respectively. In the same way, $Q\left((b_m - X_i)/h^x_{N_m}\right) = 0$ and $Q\left((d_m - Y_i)/h^y_{N_m}\right) = 0$ as $b_m \to +\infty$ and $d_m \to +\infty$ respectively. Finally as $1 - Q(x) = Q(-x)$, we get from (3.6) the expressions of $(\hat{p}_{e,N_m})_{1\leq m\leq 4}$ as follows,

$$\begin{cases} \hat{p}_{e,N_1} &= 1 - \int_0^{+\infty}\int_0^{+\infty} \hat{f}^{S_1}_{Z,N_1}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= 1 - \frac{1}{N_1}\sum\limits_{Z_i\in\mathcal{C}_1} Q\left(-\frac{X_i}{h^x_{N_1}}\right) Q\left(-\frac{Y_i}{h^y_{N_1}}\right) \\ \hat{p}_{e,N_2} &= 1 - \int_0^{+\infty}\int_{-\infty}^{0} \hat{f}^{S_2}_{Z,N_2}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= 1 - \frac{1}{N_2}\sum\limits_{Z_i\in\mathcal{C}_2} Q\left(-\frac{X_i}{h^x_{N_2}}\right) Q\left(\frac{Y_i}{h^y_{N_2}}\right) \\ \hat{p}_{e,N_3} &= 1 - \int_{-\infty}^{0}\int_{-\infty}^{0} \hat{f}^{S_3}_{Z,N_3}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= 1 - \frac{1}{N_3}\sum\limits_{Z_i\in\mathcal{C}_3} Q\left(\frac{X_i}{h^x_{N_3}}\right) Q\left(\frac{Y_i}{h^y_{N_3}}\right) \\ \hat{p}_{e,N_4} &= 1 - \int_{-\infty}^{0}\int_0^{+\infty} \hat{f}^{S_4}_{Z,N_4}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= 1 - \frac{1}{N_4}\sum\limits_{Z_i\in\mathcal{C}_4} Q\left(\frac{X_i}{h^x_{N_4}}\right) Q\left(-\frac{Y_i}{h^y_{N_4}}\right) \end{cases} \tag{4.2}$$

5. Simulation Results

5.1. Simulation Frameworks

According to (3.6), the M-QAM SEP estimation using kernel method requires a classification of the soft observations $(Z_i)_{1\leq i\leq N}$ into as many clusters $\mathcal{C}_m$ as the modulation order M. In this paper, we made the classification in a supervised fashion, i.e., the transmitted information sequence is known. The supervised classification procedure consists of the synchronisation of the received sequence positionwise with the transmitted sequence. Then, each observation Z_i of the current index is classified into the cluster assigned to its corresponding information symbol s_i of the same index.

Three types of channel models are considered: the Additive White Gaussian Noise (AWGN) channel, a frequency-selective channel and a frequency-selective Rayleigh fading channel. When the transmissions are made over the AWGN channel, the received soft observations that are used for the SEP estimation are in the form of $Z_i = s_i + \eta_i$, where η_i is a realisation of the channel noise η and $\eta \sim \mathcal{N}(0, \sigma_\eta^2)$.

The frequency-selective channel is one of the frequency-selective channel models designed in [20]. It is defined by a finite impulse response filter h with 11 coefficients. The soft observations used for the SEP estimation are taken from the channel output. They are in the form of $Z_i = \sum_{k=1}^{11} h_k s_{i-k} + \eta_i$. As this channel introduces inter-symbol interferences, an equalisation is made; afterwards the equalised soft observations are used for the SEP estimation.

The third type of channel involved in the simulation of the proposed kernel-based SEP estimator is a multi-carrier frequency-selective Rayleigh fading channel. The profile of this latter type channel is given by: a Gray-labeled 4-QAM constellations, ten taps long with a sample period of $12.8\,\mu s$, an $8\,Hz$ maximum Doppler shift and average taps gains given in $watts$ by the vector $[\,0.0616\;\;0.4813\;\;0.1511\;\;0.0320\;\;0.1323\;\;0.0205\;\;0.0079\;\;0.0778\;\;0.0166\;\;0.0188\,]$ [20], [21]. To mitigate the inter-symbol and inter-carrier interferences, we implemented an orthogonal frequency division multiplexing technique of 128 sub-carriers with a cyclic prefix of length 9 [22].

5.2. Numerical Results

The accuracy and the reliability of the proposed kernel-based SEP estimator is analysed using absolute biases and confidence intervals. The absolute bias is for evaluating how accurate is an estimate of the SEP. It is defined by $|\,\mathbb{E}[\hat{p}_{e,N}] - p_{e,N}\,|$ where $\mathbb{E}[.]$ denotes the mathematical expectation operator. The Confidence Interval (CI) is for the measure of the estimation reliability. In [3], it has been prefered to variance. To compute $\mathbb{E}[\hat{p}_{e,N}]$ and the CIs, we were provided with 101 SEP estimations trials. Based on the 101 SEP estimates values, the Student's t-distribution method was applied to determine the CIs values for a confidence level of 95%. To complete the absolute bias values calculation, we needed to be provided with values of the SEP $p_{e,N}$. So, when AWGN or Rayleigh fading channels are considered, a theoretical expression of the SEP is available. However, in the case of the frequency-selective channel, the SEP is determined in the form of a benchmark. The benchmark was computed by simulations using the standard MC method under the constraint that a threshold of one hundred errors occured. Simulations were carried out for different values of the information bit energy to noise power spectral density ratio (E_b/N_0). They covered SER values slightly greater than 10^{-1} down to small SER values: in the vicinity of 10^{-4} when the Rayleigh fading channel is involved and in the vicinity of 10^{-5} regarding the AWGN and the frequency-selective channels.

Figure 1 shows the curve of performance given by the proposed SEP estimator over the AWGN channel. Represented by the curve in solid blue line with diamond mark at each data point, the SEP estimates achieved by the proposed estimator are compared to the theoretical SEP values borne by the curve in red solid line. Following the same legend (i.e., solid blue line with diamond mark at each data point for the proposed estimator and solid red line for the reference curve), Figure 2 and Figure 3 also show the estimation performance achieved over the frequency-selective and the Rayleigh fading channels respectively. These three figures have in common that their plots allow the proposed SEP estimator accuracy to be visually evaluated. The observed SEP estimates as borne by Figure 1, Figure 2 and Figure 3 correspond to the absolute bias numerical data given in the third columns of Table 1, Table 2 and Table 3 respectively. As these tables let us see, the absolute bias values are negligible compared to the true values $p_{e,N}$. This explain why, as the curves in the figures look like, quite all the SEP estimates data points appear pointwise consistent. The resulting accuracy of the SEP estimations is therefore very high.

Beyond the demonstrated high accuracy of the SEP estimations, let us now analyse their reliability. For this, the CIs values characterising the achieved reliability are given in the fourth columns of Table 1, Table 2 and Table 3. The smaller the CI is, more reliable is the SEP estimate. Throughout Table 1, Table 2 and Table 3, the achieved CIs are very small: most of them are close to $[0.9p_{e,N},\ 1.1p_{e,N}]$ and the largest CI is given by $[0.84p_{e,N},\ 1.16p_{e,N}]$. So, the proposed kernel-based SEP estimator is highly reliable under the three types of channel models.

The performance achieved in terms of accuracy and reliability by the proposed kernel-based SEP estimator is at the cost of a certain sample size N_K, i.e., the amount of soft observations that

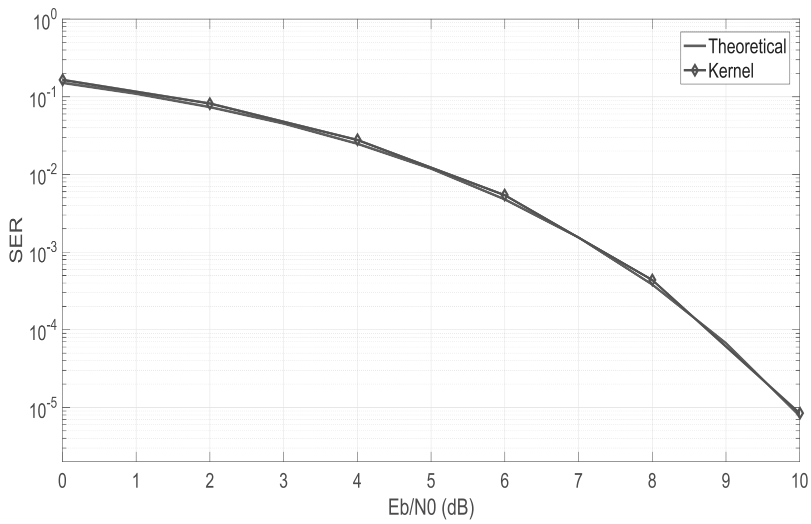

Figure 1: 4-QAM SEP estimates over the AWGN channel.

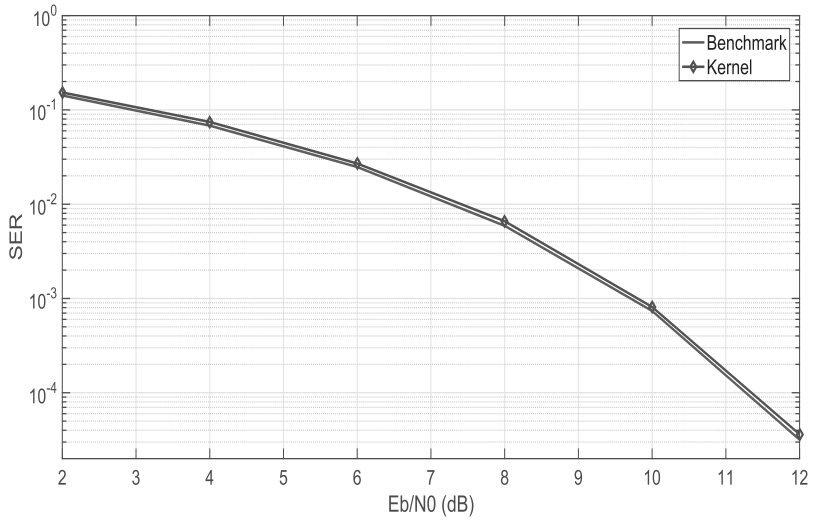

Figure 2: 4-QAM SEP estimates over the the frequency-selective channel.

were used for the estimation. In the fith columns of Table 1, Table 2 and Table 3, the different

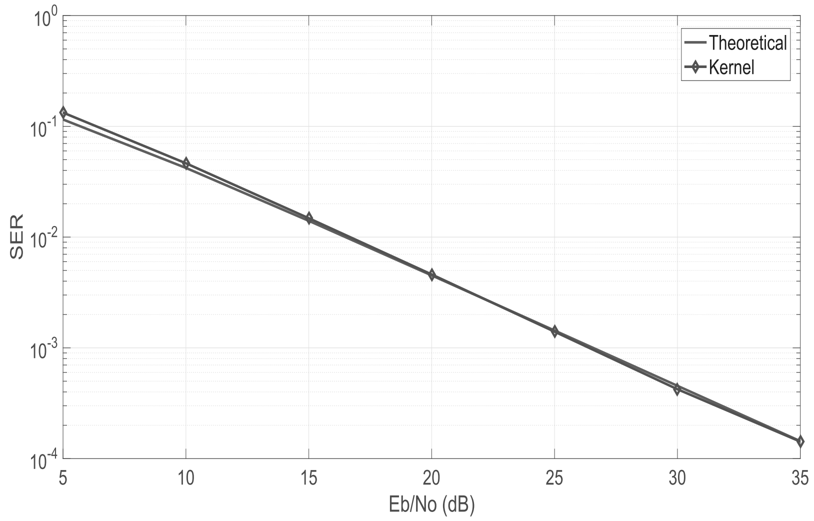

Figure 3: 4-QAM SEP estimates over the Rayleigh channel.

$E_b/N_0(dB)$	$p_{e,N}$	Absolute bias	Two-sided CI	N_K	N_{mc}
00	1.51×10^{-1}	0.14×10^{-1}	$[0.99p_{e,N},\ 1.01p_{e,N}]$	3.0×10^3	4.5×10^3
02	7.36×10^{-2}	0.83×10^{-2}	$[0.99p_{e,N},\ 1.01p_{e,N}]$	6.0×10^3	9.0×10^3
04	2.48×10^{-2}	0.30×10^{-2}	$[0.99p_{e,N},\ 1.01p_{e,N}]$	1.5×10^4	3.5×10^4
06	4.80×10^{-3}	0.60×10^{-3}	$[0.99p_{e,N},\ 1.01p_{e,N}]$	3.5×10^4	5.5×10^4
08	3.82×10^{-4}	0.56×10^{-4}	$[0.97p_{e,N},\ 1.03p_{e,N}]$	1.0×10^5	1.5×10^5
10	7.74×10^{-6}	0.67×10^{-6}	$[0.85p_{e,N},\ 1.15p_{e,N}]$	2.0×10^5	2.3×10^5

Table 1: Numerical data characterising the SEP estimates over the AWGN channel.

sample sizes N_K that characterise the achieved performance are given. In the last columns of Table 1, Table 2 and Table 3, numerical data corresponding to the sample sizes N_{mc} that are required by the standard MC to achieve the SEP estimations with almost equal reliability and accuracy are also given. The values of N_{mc} have been determined by simulations. Compared to the sample sizes N_{mc}, the sample sizes N_K are smaller. This results in sample size savings making the proposed SEP estimator more efficient than the standard MC. However, we may wonder if the achieved sample size savings would be enough to declare the proposed estimator efficient indeed. To answer this query, let us analyse the efficiency of the estimation in terms of CPU time. Based on the numerical data regarding the SEP estimations over the frequency-selective Rayleigh fading

$E_b/N_0(dB)$	$p_{e,N}$	Absolute bias	Two-sided CI	N_K	N_{mc}
02	1.42×10^{-1}	0.11×10^{-1}	$[0.99p_{e,N},\ 1.01p_{e,N}]$	5.0×10^3	9.0×10^3
04	6.81×10^{-2}	0.64×10^{-2}	$[0.99p_{e,N},\ 1.01p_{e,N}]$	1.0×10^4	1.5×10^4
06	2.48×10^{-2}	0.22×10^{-2}	$[0.99p_{e,N},\ 1.01p_{e,N}]$	3.0×10^4	4.0×10^4
08	5.90×10^{-3}	0.70×10^{-3}	$[0.99p_{e,N},\ 1.01p_{e,N}]$	5.0×10^4	7.0×10^4
10	7.37×10^{-4}	0.76×10^{-4}	$[0.98p_{e,N},\ 1.02p_{e,N}]$	1.0×10^5	1.2×10^5
12	3.18×10^{-5}	0.45×10^{-5}	$[0.90p_{e,N},\ 1.10p_{e,N}]$	1.0×10^5	1.6×10^5

Table 2: Numerical data characterising the SEP estimates over the frequency-selective channel.

$E_b/N_0(dB)$	$p_{e,N}$	Absolute bias	Two-sided CI	N_K	N_{mc}
05	1.15×10^{-1}	0.18×10^{-1}	$[0.92p_{e,N},\ 1.08p_{e,N}]$	3.0×10^4	1.0×10^4
10	4.21×10^{-2}	0.43×10^{-2}	$[0.90p_{e,N},\ 1.10p_{e,N}]$	3.0×10^4	6.0×10^4
15	1.40×10^{-2}	0.08×10^{-2}	$[0.88p_{e,N},\ 1.12p_{e,N}]$	3.0×10^4	6.0×10^4
20	4.50×10^{-3}	0.10×10^{-3}	$[0.87p_{e,N},\ 1.13p_{e,N}]$	3.0×10^4	6.0×10^4
25	1.40×10^{-3}	0.00×10^{-3}	$[0.86p_{e,N},\ 1.14p_{e,N}]$	3.0×10^4	7.0×10^4
30	4.54×10^{-4}	0.31×10^{-4}	$[0.84p_{e,N},\ 1.16p_{e,N}]$	3.0×10^4	8.0×10^4
35	1.44×10^{-4}	0.01×10^{-4}	$[0.84p_{e,N},\ 1.16p_{e,N}]$	6.0×10^4	1.3×10^5

Table 3: Numerical data characterising the SEP estimates over the Rayleigh channel.

channel, we observed gains of efficiency in terms of the computational cost. For instance, we noted for $E_b/N_0 = 30\,dB$ (see the corresponding row in Table 3) that the proposed SEP estimator performed using a sample size $N_K = 3.0 \times 10^4$ and engendering a computational cost (i.e., the CPU time) of $2.0\,s$. In the same conditions of transmission and achieving the same accuracy and reliability performance, the standard MC method used a sample size $N_{mc} = 8.0 \times 10^4$ causing a higher CPU time equal to $4.89\,s$. A second illustration, based on the data of the last row of Table 3, reveals a CPU time of $3.64\,s$ against $7.87\,s$ for the standard MC method. CPU time savings are therefore achieved. These CPU time savings, although they seem small, may be of a great interest in green computing.

The asymptotic behaviour of the proposed kernel-based SEP estimator is illustrated in Table 4 for transmissions over the frequency-selective channel under two different E_b/N_0 values. We can see that the absolute bias and the CI become smaller as the sample size N_K increases.

$E_b/N_0(dB)$	N_K	Absolute bias	Two-sided CI
02	$3,000$	0.14×10^{-1}	$[0.989p_{e,N},\ 1.011p_{e,N}]$
	$4,000$	0.12×10^{-1}	$[0.992p_{e,N},\ 1.008p_{e,N}]$
	$5,000$	0.11×10^{-1}	$[0.993p_{e,N},\ 1.007p_{e,N}]$
	$6,000$	0.10×10^{-1}	$[0.993p_{e,N},\ 1.007p_{e,N}]$
12	$5,000$	2.35×10^{-5}	$[0.573p_{e,N},\ 1.427p_{e,N}]$
	$10,000$	2.11×10^{-5}	$[0.601p_{e,N},\ 1.399p_{e,N}]$
	$50,000$	0.86×10^{-5}	$[0.862p_{e,N},\ 1.138p_{e,N}]$
	$100,000$	0.45×10^{-5}	$[0.903p_{e,N},\ 1.097p_{e,N}]$

Table 4: Asymptotic behaviour of the proposed SEP estimator.

6. Conclusion

In this paper, we investigated the problem of efficient Symbol Error Probability (SEP) estimation for the transmission systems based on Quadrature Amplitude Modulation (QAM) schemes. We designed a Gaussian kernel-based SEP estimator that allows, using soft observations at the receiver-end, to compute the Symbol Error Rate of any digital communication system built over a QAM transceiver. We first established the formal expression of the SEP and presented the way it can be estimated using kernel density estimators. Then, we applied a two-dimensional kernel density estimators technique to obtain a convenient equation of the SEP estimate that can be computed by simulation. We used soft 4-QAM symbols sampled from the channel output and run simulations under three different types of transmission channels: the Additive White Gaussian Noise channel, a frequency-selective channel and a multipath Rayleigh fading channel. We reported simulation results of the kernel-based SEP estimates and analysed, using absolute bias and confidence interval, the performance of the estimator in terms of accuracy and reliability. For equivalent accuracy and reliability, we compared the proposed kernel-based SEP estimator to the standard Monte Carlo simulation technique and found that the proposed estimator is more efficient. We evaluated its efficiency on both levels of sample size and CPU time savings. The overall performance of the proposed kernel-based estimator was concluded highly accurate and reliable to the trade-off of an efficiency up to a factor two. The achieved efficiency of the kernel-based estimator can be of greater significance in some issues resolution that are better addressed by the green computing research area.

References

[1] S. Benedetto, E. Biglieri, R. Daffara. *Modeling and performance evaluation of non linear satellite links-A volterra series approach*. IEEE Journal on Selected Areas in Communications, AES-15 (1979), 494-507.

[2] K. Yao, L. B. Milstein. *The use of moment space bounds for evaluating the performance of a non linear digital communication system*. IEEE Transactions on Communications, 31 (1993), 677-683.

[3] C. M. Jeruchim. *Techniques for estimating the bit error rate in the simulation of digital communication systems*. IEEE Journal on Selected Areas in Communications, 2 (1984), No. 1, 153-170.

[4] K. Shanmugan, P. Balaban. *A Modified Monte-Carlo Simulation Technique for the Evaluation of Error Rate in Digital Communication Systems*. IEEE Transactions on Communications, 28 (1980), No. 11, 1916-1924.

[5] S. B. Weinstein. *Theory and application of some classical and generalized asymptotic distributions of extreme values*. IEEE Transactions on Information Theory, 19 (1973), No. 2, 148-154.

[6] S. B. Weinstein. *Estimation of small probabilities by linearization of the tail of a probability distribution function*. IEEE Transactions on Communication Technology, 19 (1971), No. 6, 1149-1155.

[7] M. Rosenblatt. *Remarks on some non-parametric estimates of a density function*. The Annals of Mathematical Statistics, 27 (1956), No. 3, 832-837.

[8] S. Saoudi, F. Ghorbel and A. Hillion. *Some statistical properties of the kernel-diffeomorphism estimator*. Applied Stochastic Models and Data Analysis Journal, 13 (1997), 39-58.

[9] S. Saoudi, M. Troudi, F. Ghorbel. *An Iterative Soft Bit Error Rate Estimation of Any Digital Communication Systems Using a Nonparametric Probability Density Function*. EURASIP Journal on Wireless Communications and Networking, (2009), doi:10.1155/2009/512192.

[10] S. Saoudi, T. Ait-Idir, Y. Mochida. *A Novel Non-Parametric Iterative Soft Bit Error Rate Estimation Technique for Digital Communications Systems*. IEEE International Conference on Communications (ICC), (2011), 1-6.

[11] J. D. Laster, J. H. Reed, W. H. Tranter. *Bit Error Rate Estimation using Probability Density Function Estimators*. IEEE Transactions on Vehicular Technology, 51 (2003), No. 1, 260-267.

[12] P. Poda, S. Saoudi, T. Chonavel, F. Guilloud, T. Tapsoba. *Kernel-based performance evaluation of coded QAM systems*. CARI 2016, Hammamet, Tunisie, (2016), 182-191.

[13] P. Poda, T. Tapsoba, S. Saoudi, T. Chonavel, F. Guilloud. *A kernel-based soft BER estimator for coded QAM transmission systems*. IEEE International Symposium on Signal, Image, Video and Communications (ISIVC), (2016), 314-319.

[14] P. Poda, S. Saoudi, T. Chonavel. *An efficient Kernel-Based Technique for QAM Symbol Error Probability Estimation*. IEEE Wireless Communications and Networking Conference Workshops (WCNCW), (2017), 1-6.

[15] P. Poda, S. Saoudi. *A kernel-based QAM symbol error probability estimation Technique*. IEEE Vehicular Technology Conference (VTC-Fall), (2017), Toronto, Canada (accepted).

[16] I. Kom, L. M. Hii. *Relation Between Bit and Symbol Error Probabilities for DPSK and FSK with Differential Phase Detection*. IEEE Transactions on Commununications, 42 (1994), No. 10, 2778-2780.

[17] Z. Z. Adriano, D. Ronaldo. *A review of kernel density estimation with application to econometrics*. International Econometric Review, 5 (2013), 20-42.

[18] K. S. Shanmugan, A. M. Breipohl. Random Signals: Detection, Estimation, and Data Analysis. Whiley, New York, 1988.

[19] M. Troudi, A. M. Alimi, S. Saoudi. *Fast plug-in method for parzen probability density estimator applied to genetic neutrality study*. International Conference on Computer as a Tool (EUROCON'07), (2007), 1034-1039.

[20] J. G. Proakis, M. Salehi. Digital Communications. Mc Graw Hill, 5th edition, New York, 2008.

[21] T. S. Rappaport. Wireless Communications principles and practice. New Jersey: Prentice Hall PTR, 1996.

[22] J. R. Barry, E. A. Lee, D. G. Messerschmitt. Digital Communication. Kluwer Academic Publishers, 3rd edition, 2004.