Threshold ages for the relation between lifetime entropy and mortality risk

Patrick Meyer, Gregory Ponthiere

Threshold Ages for the Relation between Lifetime Entropy and Mortality Risk*

Patrick Meyer† Gregory Ponthiere‡

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Abstract

We study the effect of a change in age-specific probability of death on risk about the duration of life measured by Shannon's entropy defined to the base 2. We first show that a rise in the probability of death at age n increases lifetime entropy at age $k \leq n$ if and only if the quantity of information revealed by the event of a death at age n exceeds lifetime entropy at age $n+1$ divided by the probability to survive from age k to age $n+1$. There exist, under general conditions, two threshold ages: first, a low threshold age below which a rise in mortality risk decreases lifetime entropy, and above which it raises lifetime entropy; second, a high threshold age above which a rise in mortality risk reduces lifetime entropy. Using French life tables, we show that the gap between those two threshold ages has been increasing over the last two centuries.

Keywords: mortality risk, lifetime entropy, threshold age.
JEL classification codes: J10, D83.

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†IMT Atlantique, Lab-STICC, Université Bretagne Loire.
‡University Paris Est (ERUDITE), Paris School of Economics and Institut universitaire de France. [corresponding author]. Address: Paris School of Economics, 48 boulevard Jourdan, 75014 Paris, France. Telephone: 0033-0180521919. E-mail: gregory.ponthiere@ens.fr

1 Introduction

Risk about the duration of life is a major dimension of human condition. Whereas all individuals know that they will die one day, no one knows precisely *when* one's death will take place. This risk is faced by everyone, but is somewhat diffuse and abstract, and thus hard to quantify in an intuitive way.[1]

In a recent work, Meyer and Ponthiere (2019) proposed to quantify risk about the duration of life by means of Shannon's entropy index defined to the base 2 (see Shannon 1948). Lifetime entropy is defined as follows:[2]

$$H_k = -\sum_{i=k}^{115} p_{i,k} \log_2 (p_{i,k}) \tag{1}$$

where H_k denotes lifetime entropy at age k and $p_{i,k}$ is the probability of a life of length i for an individual of age k.

Lifetime entropy H_k is the mathematical expectation of the quantity of information revealed by the event of a life of a particular length $i \geq k$, or, alternatively, of the quantity of information revealed by the event of a death at an age $i \geq k$. As such, H_k can be regarded as the informational equivalent of the standard life expectancy. Instead of measuring the mathematical expectation of the duration of life, H_k measures the mathematical expectation of the quantity of information revealed by a particular duration of life.

A specificity of Shannon's lifetime entropy with respect to other measures of risk about the duration of life, such as Kannisto's coefficient ${}_{10}C_{50}$ (Kannisto 2000), the standard deviation of the age at death (Lan Karen Cheung and Robine 2007, Edwards and Tuljapurkar 2005) and Gini indexes of the length of life (Smits and Monden 2009), is to measure risk about the duration of life in terms of bits, i.e. the quantity of information revealed by tossing a fair coin. As such, that indicator makes the - quite abstract - risk about the duration of life commensurable with the risk involved in tossing a given number of fair coins, a life experience with which individuals are familiar.[3] Thus, thanks to its reliance on the bit metric, Shannon's lifetime entropy index allows to make humans more familiar with the risk about the duration of life, by making that risk comparable with the risk involved in tossing fair coins.

An interesting application of Shannon's lifetime entropy index consists of computing that H_k at various ages of life $k, k+1, \ldots$. The resulting series $H_k, H_{k+1}, \ldots$ shows how the risk about the duration of life is progressively resolved when individuals become older and older.[4] In young adulthood, numerous scenarios can occur concerning the duration of life, and lifetime entropy is

[1] On the various measures of risk about the duration of life, see Wilmoth and Horiuchi (1999) and Van Raalte and Caswell (2013).

[2] That lifetime entropy index is close to the one used in Hill (1993) and Noymer and Coleman (2014), but differs regarding the basis: we use base 2, whereas Hill (1993) uses base e and Noymer and Coleman (2014) use base 10.

[3] See Lanciani (1892) on tossing fair coins under the Roman Empire.

[4] See Meyer and Ponthiere (2019).

high. On the contrary, once old ages are reached, the number of possible scenarios for the duration of life becomes smaller, and lifetime entropy is reduced. Thus the computation of Shannon's lifetime entropy indexes at successive ages provides also a picture of how the quantity of risk about the duration of life varies with the age, and is progressively resolved when individuals become older.

Whereas the effect of a change of an age-specific probability of death on standard life expectancy is unambiguous, the same is not true as far as its effect on its informational equivalent - i.e. human lifetime entropy - is concerned. Hence an interesting question is to know how sensitive lifetime entropy is to a change in age-specific probability of death. Does a rise in mortality risk at a given age tend to increase, or, on the contrary, to decrease risk about the duration of life measured by lifetime entropy?

Studying the effect of age-specific mortality on lifetime entropy matters for several reasons. First, when there is a change in an age-specific probability of death, this change affects the entire probability distribution of life durations in a non-trivial way: whereas some durations of life become more likely than before the change, other scenarios, associated to other durations of life, become less likely. Hence the net effect on lifetime entropy is hard to quantify. Second, individual decisions in terms of insurance or precautionary savings depend not only on the expected duration of life, but, also, on the extent of risk about the duration of life.[5] Hence, it is important, from a predictive perspective, to understand how mortality shocks affect not only life expectancy, but, also, lifetime entropy. Third, in real-world economies, mortality shocks due to, for instance, epidemics, heat waves or pollution peaks, can affect various ages of life in distinct ways, and there is *a priori* no reason why the effect of a mortality shock on lifetime entropy would be of the same sign or magnitude across the different ages of life. In order to better understand the impact of those mortality shocks on lifetime entropy, it is thus important to examine how the relation between a mortality shock and lifetime entropy varies with age.

The goal of this paper is to examine the impact of a change in age-specific mortality risk on risk about the duration of life, as measured by Shannon's lifetime entropy index. For that purpose, we first propose to decompose that effect into its various components, to identify the necessary and sufficient condition under which a rise in mortality risk at a given age contributes to increase lifetime entropy. Then, we use that condition to study how the sign of the effect of a change in mortality on lifetime entropy varies with age.

Anticipating our results, we first show that a rise in the probability of death at age n increases lifetime entropy at age $k \leq n$ if and only if the quantity of information revealed by the event of a death at age n exceeds lifetime entropy at age $n+1$ divided by the probability to survive from age k to age $n+1$. Then, we show that there exist, under general conditions, two threshold ages: first, a low threshold age below which a rise in mortality risk decreases lifetime entropy, and above which it raises lifetime entropy; second, a high threshold age above

[5] The reason why the amount of risk about the duration of life matters is simply that individuals are generally not risk-neutral with respect to the duration of life, but, on the contrary, are generally risk-averse with respect to the duration of life (see Bommier 2006).

which a rise in mortality risk reduces lifetime entropy. Finally, using French life tables, we identify those two threshold ages, and we show that the gap between these has been increasing over the last two centuries.

Our work is related to Zhang and Vaupel (2009), which examines the effect of averting deaths on life disparity measured by life expectancy lost due to death. Using a framework with age as a continuous variable, Zhang and Vaupel (2009) showed that there exist, under general conditions on Keyfitz's entropy of the life table (Keyfitz 1977), a unique threshold age below which averting deaths reduces life disparity, and above which averting deaths increases life disparity. The existence and characterization of that unique threshold age is also studied recently by Aburto et al (2019). In comparison to Zhang and Vaupel (2009) and Aburto et al (2019), our approach differs on three main grounds. First, we focus on the effect of a change in mortality on risk about the duration of life not measured by means of lifetime disparity as measured by life expectancy lost due to death, but by means of Shannon's lifetime entropy index defined to the base 2. Second, at the technical level, our framework is in discrete time rather than in continuous time, which makes the identification of threshold ages more difficult to prove analytically. Third, at the level of results, whereas Zhang and Vaupel (2009) and Aburto et al (2019) identify a *unique* threshold age using conditions on life table entropy, we identify, on the contrary, not one, but *two* threshold ages, by making assumptions on the pattern of age-specific mortality. The high threshold age (below which a rise in mortality raises lifetime entropy, and beyond which it reduces lifetime entropy) is quite similar to the threshold age studied by Zhang and Vaupel (2009) and Aburto et al (2019). In addition, we show that there exists also another, lower threshold age, below which a rise in mortality reduces lifetime entropy. We show that this low threshold age, which was equal to 6 years in the early 19th century, has turned out to vanish to age 0 in the second half of the 20th century, leaving us with a unique threshold age, below which a rise in mortality raises lifetime entropy, and above which a rise in mortality reduces lifetime entropy.

The rest of the paper is organized as follows. Section 2 studies the relation between Shannon's lifetime entropy and the probability of death at a particular age. Then, Section 3 examines the existence of threshold ages at which the sign of that relation changes. Our findings are illustrated by means of the case of France (1816-2016) in Section 4. Concluding remarks are left to Section 5.

2 Relationship

We consider a discrete time model where age is a natural number between the minimum age 0 and the maximum age $M > 0$. In that model, durations of life are also natural numbers, and go from 0 to M life-periods. As in any discrete time model, events can, by definition, only take place at some points in time. For simplicity, it is assumed that, when the event of a death takes place during a life-period, it takes place at the very beginning of that life-period, that is, the duration of life for a person who dies at an age $k \geq 0$ is exactly equal to k

periods. In the rest of this paper, we denote the probability of death at age k conditionally on survival to age k by d_k.

We measure risk about the duration of life by means of Shannon's lifetime entropy index defined to the base 2. Using the identity $p_{i,k} \equiv s_{i,k} d_i$, where $s_{i,k} = \prod_{j=k}^{i-1} (1 - d_j)$ is the probability of survival to age i for an individual of age k, Shannon's lifetime entropy index at age k can be rewritten as:[6]

$$H_k = -\sum_{i=k}^{M} \left(\prod_{j=k}^{i-1} (1 - d_j) d_i \right) \log_2 \left(\prod_{j=k}^{i-1} (1 - d_j) d_i \right) \tag{2}$$

Shannon's lifetime entropy at age k measures the mathematical expectation of the quantity of information revealed by the event of a death at a particular age $i \geq k$.

Quite interestingly, Shannon's lifetime entropy index can be decomposed in terms of Wiener's entropy indexes, as follows:[7]

$$H_k = \sum_{i=k}^{M} \left(\prod_{j=k}^{i-1} (1 - d_j) d_i \right) \left[W \left(\prod_{j=k}^{i-1} (1 - d_j) \right) + W (d_i) \right] \tag{3}$$

where $W(x) = -\log_2(x)$ denotes Wiener's entropy (Wiener (1948), which captures the quantity of information revealed by the occurrence of a single event with probability x.

The above decomposition states that Shannon's lifetime entropy at age k is the mathematical expectation of the quantity of information revealed by the event of a death at age $i \geq k$, which is composed of two quantities of information: on the one hand, the quantity of information revealed by the event "survival to age i" (first term in brackets, i.e. $W(s_{i,k})$), and, on the other hand, the quantity of information revealed by the event "death at age i conditionally on survival to age i" (second term in brackets, i.e. $W(d_i)$).

The decomposition of Shannon's lifetime entropy index into Wiener's entropy indexes is not only valuable *per se*, as a finer description of the components of risk about the duration of life. As we shall see below, that decomposition allows us also to be able to cast new light on the relation between lifetime entropy and the risk of death at a particular age, that is, the relation between the risk about the duration of life and the risk of death at a particular age.

To study this, let us now examine the effect of a variation of the probability of death d_n on lifetime entropy at age $k \leq n$. Our results are summarized in Proposition 1, which states the necessary and sufficient condition under which a rise of the probability of death d_n increases lifetime entropy at age k.

[6] Note that the probability $s_{i,k}$ is distinct from the probability $p_{i,k}$. The probability $s_{i,k}$ is the probability, for a person of age k, to survive up to - at least - age i, and, thus, is the likelihood of a life of a length equal to either i, or $i+1$, ..., or M. The probability $p_{i,k}$ measures the probability, for a person of age k, of a life of *exact* length i. Thus we obviously have: $p_{i,k} \equiv s_{i,k} d_i < s_{i,k}$.

[7] On this decomposition, see Meyer and Ponthiere (2019).

Proposition 1 *A rise in the probability d_n increases (resp. decreases) lifetime entropy at age k if and only if:*

$$W(s_{n,k}) + W(d_n) > \text{ (resp. } < \text{)} \frac{H_{n+1}}{s_{n+1,k}}$$

where $W(x) = -\log_2(x)$ is Wiener's entropy (Wiener 1948).

Proof. In order to study the effect of a change of the probability d_n on Shannon's lifetime entropy at age k, we can first use the decomposition of H_k in terms of Wiener's entropy indexes, to rewrite H_k as follows:

$$\begin{aligned} H_k &= \sum_{i=k}^{n-1}\left(\prod_{j=k}^{i-1}(1-d_j)d_i\right)\left[W\left(\prod_{j=k}^{i-1}(1-d_j)\right)+W(d_i)\right] \\ &+ \left(\prod_{j=k}^{n-1}(1-d_j)d_n\right)\left[W\left(\prod_{j=k}^{n-1}(1-d_j)\right)+W(d_n)\right] \\ &+ \sum_{i=n+1}^{M}\left(\prod_{j=k}^{i-1}(1-d_j)d_i\right)\left[W\left(\prod_{j=k}^{i-1}(1-d_j)\right)+W(d_i)\right] \end{aligned} \tag{4}$$

Using that decomposition, we can compute the derivative $\frac{\partial H_k}{\partial d_n}$ as:

$$\begin{aligned} \frac{\partial H_k}{\partial d_n} &= \left(\prod_{j=k}^{n-1}(1-d_j)\right)\left[W\left(\prod_{j=k}^{n-1}(1-d_j)\right)+W(d_n)\right] + \left(\prod_{j=k}^{n-1}(1-d_j)d_n\right)W'(d_n) \\ &+ \sum_{i=n+1}^{M}\left(-\prod_{j=k\backslash n}^{i-1}(1-d_j)d_i\right)\left[W\left(\prod_{j=k}^{i-1}(1-d_j)\right)+W(d_i)\right] \\ &+ \sum_{i=n+1}^{M}\left(\prod_{j=k}^{i-1}(1-d_j)d_i\right)\left[W'\left(\prod_{j=k}^{i-1}(1-d_j)\right)\left(-\prod_{j=k\backslash n}^{i-1}(1-d_j)\right)\right] \end{aligned}$$

Hence, we have:

$$\begin{aligned} \frac{\partial H_k}{\partial d_n} &= \underbrace{s_{n,k}\left[W(s_{n,k})+W(d_n)\right]}_{\text{effect of } \Delta d_n \text{ on likelihood of life of length } n \text{ (+)}} \\ &+ \underbrace{s_{n,k}d_nW'(d_n)}_{\text{effect of } \Delta d_n \text{ on information revelation of life of length } n \text{ (-)}} \\ & \underbrace{-\frac{1}{1-d_n}H_{n+1}}_{\text{effect of } \Delta d_n \text{ on likelihood of life of length } > n \text{ (-)}} \\ &+ \sum_{i=n+1}^{M} \underbrace{(s_{i,k}d_i)\left[W'(s_{i,k})\left(-\frac{s_{i,k}}{(1-d_n)}\right)\right]}_{\text{effect of } \Delta d_n \text{ on information revelation of life of length } > n \text{ (+)}} \end{aligned}$$

where $W'(x) = \frac{-1}{x\ln(2)} < 0$.

Note that this expression can be simplified further. Indeed, the second term can be rewritten as:

$$s_{n,k} d_n W'(d_n) = s_{n,k} d_n \frac{-1}{d_n \ln(2)} = s_{n,k} \frac{-1}{\ln(2)} \tag{5}$$

while the fourth term can be rewritten as:

$$\sum_{i=n+1}^{M} (s_{i,k} d_i) \left[W'(s_{i,k}) \left(-\frac{s_{i,k}}{(1-d_n)} \right) \right] = \frac{1}{\ln(2)} \frac{1}{(1-d_n)} \underbrace{\sum_{i=n+1}^{M} (s_{i,k} d_i)}_{=s_{n,k}(1-d_n)} = \frac{s_{n,k}}{\ln(2)} \tag{6}$$

since $1 = \sum_{i=k}^{n} s_{i,k} d_i + \sum_{i=n+1}^{M} s_{i,k} d_i \iff \sum_{i=n+1}^{M} s_{i,k} d_i = 1 - \sum_{i=k}^{n} s_{i,k} d_i = s_{n,k}(1-d_n)$.

Thus the second and the fourth terms of $\frac{\partial H_k}{\partial d_n}$ cancel out. We thus have:

$$\begin{aligned} \frac{\partial H_k}{\partial d_n} \quad \gtrless \quad 0 &\iff s_{n,k} \left[W(s_{n,k}) + W(d_n) \right] - \frac{1}{1-d_n} H_{n+1} \gtrless 0 \\ &\iff W(s_{n,k}) + W(d_n) \gtrless \frac{H_{n+1}}{s_{n+1,k}} \end{aligned} \tag{7}$$

■

Proposition 1 states that a rise in the probability of death d_n increases lifetime entropy at age k if and only if the quantity of information revealed by the event of a death at age n (measured by means of Wiener's entropy) exceeds Shannon's lifetime entropy at age $n+1$ divided by the probability to survive from age k to age $n+1$.

3 Existence of threshold ages

The condition stated in Proposition 1 can be used to show that there exist, under general conditions on the age-pattern of mortality, two threshold ages, around which the sign of the derivative $\frac{\partial H_k}{\partial d_n}$ varies. For that purpose, this section will focus on the variation of lifetime entropy at birth, that is, on how the age affects the sign of the derivative $\frac{\partial H_0}{\partial d_n}$.

The study of the existence of threshold ages at which the sign of $\frac{\partial H_0}{\partial d_n}$ changes requires to impose some assumptions on how mortality risks vary with age. Throughout this section, we assume that the risk of death d_n is first decreasing with age, and, then, increasing with age. That assumption is quite standard in the demographic literature, since it is well-known that the very beginning of life is characterized by a higher mortality risk, which then falls during childhood, before becoming increasing during the adult period (see Preston et al 2001). The relation between age and mortality risk exhibits thus a U-shaped pattern.

The results of our calculations are summarized in Proposition 2.

Proposition 2 *Assume that the risk of death is first decreasing with age during childhood, and, then, increasing with age during the rest of life. There exist two threshold ages a_1 and a_2 with $0 \leq a_1 < a_2$ such that:*

- *when $n < a_1$, a rise in d_n decreases lifetime entropy at birth H_0;*
- *when $a_1 < n < a_2$, a rise in d_n increases lifetime entropy at birth H_0;*
- *when $a_2 < n$, a rise in d_n decreases lifetime entropy at birth H_0.*

Proof. Let us first prove the existence and uniqueness of the low threshold age $a_1 \geq 0$.

At age 0 ($n = 0$), the condition of Proposition 1 becomes:

$$\frac{\partial H_0}{\partial d_0} \gtreqless 0 \iff W(d_0) \gtreqless \frac{H_1}{(1 - d_0)} \tag{8}$$

The higher d_0 is, and the lower the left-hand side (LHS) is, while the higher the right-hand side (RHS) is. We can show that there exists a threshold for infant mortality $\bar{d}_0$ such that for $d_0 > \bar{d}_0$ we have $\frac{\partial H_0}{\partial d_0} < 0$ and for $d_0 < \bar{d}_0$ we have $\frac{\partial H_0}{\partial d_0} > 0$.

Assume that d_0 tends to 0. Then the LHS of the above condition $W(d_0) = -\log_2(d_0)$ tends to $+\infty$, while the RHS tends to H_1, which is finite, thus the LHS exceeds the RHS, implying that $\frac{\partial H_0}{\partial d_0} > 0$. Assume, on the contrary, that d_0 tends 1. Then the LHS tends to 0, while the RHS tends to $+\infty$. This implies that the RHS exceeds the LHS, implying that $\frac{\partial H_0}{\partial d_0} < 0$.

By continuity, there exists a threshold of infant mortality $\bar{d}_0$ in $]0, 1[$ such that $W(\bar{d}_0) = \frac{H_1}{(1-\bar{d}_0)}$, at which a marginal change in infant mortality has no impact on lifetime entropy, i.e. $\frac{\partial H_0}{\partial d_0} = 0$. That threshold is unique, since the LHS is strictly decreasing in d_0, while the RHS is strictly increasing in d_0.

Hence, when infant mortality is higher than the threshold $\bar{d}_0$ (which depends on H_1), a rise in infant mortality reduces lifetime entropy, i.e. $\frac{\partial H_0}{\partial d_0} < 0$. When it is lower than the threshold, it raises lifetime entropy, i.e. $\frac{\partial H_0}{\partial d_0} > 0$.

With age, mortality falls during childhood and then rises during adulthood. Consider first childhood. The condition at age $n > 0$ becomes:

$$\frac{\partial H_0}{\partial d_n} \gtreqless 0 \iff W(s_{n,0}) + W(d_n) \gtreqless \frac{H_{n+1}}{s_{n+1,0}} \tag{9}$$

By substituting for H_{n+1}, that condition can be rewritten as:

$$\frac{\partial H_0}{\partial d_n} \gtreqless 0 \iff \frac{1}{(1-d_n)^2}\left[W(d_n) - H_n\right] \gtreqless -W(1-d_n) + \frac{1}{(1-d_n)}\left[W(d_n)(1 - s_{n+1,0}) - s_{n+1,0} W(s_{n,0})\right] \tag{10}$$

Note that, during childhood, we have that n is very low. As a consequence, $W(s_{n,0})$ is close to 0 (since $s_{n,0}$ is close to 1), and $s_{n+1,0}$ is close to 1, so that the condition can be approximated by:

$$\frac{\partial H_0}{\partial d_n} \gtrless 0 \iff \frac{1}{(1-d_n)^2}\left[W(d_n) - H_n\right] \gtrless -W(1-d_n) \tag{11}$$

In the light of that condition, two cases can arise during childhood.

If $d_0 < \bar{d}_0$ and d_n remains low during childhood, the quantity of information revealed by the event of a death at a childhood age n, i.e. $W(d_n)$, remains larger than the mathematical expectation of the amount of information revealed by the event of a death at an age $i \geq n$, i.e. H_n, so that $W(d_n) > H_n$, implying $\frac{\partial H_0}{\partial d_n} > 0$ during childhood.

If $d_0 > \bar{d}_0$ and d_n remains high during childhood, we have that $W(d_n) < H_n$ prevails during childhood. In that case, one can have either $\frac{\partial H_0}{\partial d_n} < 0$ or $\frac{\partial H_0}{\partial d_n} > 0$ for childhood ages $n > 0$. Note, however, that since mortality risk is falling with age during childhood, sooner or later one will reach an age such that the quantity of information revealed by the event of a death at that age becomes larger than the mathematical expectation of the amount of information revealed by the event of a death at a future age, so that $W(d_n) > H_n$, implying $\frac{\partial H_0}{\partial d_n} > 0$.

Consider now young adult mortality, at which d_n is very low and increasing with age. Let us turn back to the general formula for $\frac{\partial H_0}{\partial d_n}$:

$$\begin{aligned}\frac{\partial H_0}{\partial d_n} \gtrless 0 \quad \iff \quad & \frac{1}{(1-d_n)^2}\left[W(d_n) - H_n\right] \gtrless \\ & -W(1-d_n) + \frac{1}{(1-d_n)}\left[W(d_n)(1-s_{n+1,0}) - s_{n+1,0}W(s_{n,0})\right]\end{aligned} \tag{12}$$

That formula can be rewritten as:

$$\begin{aligned}\frac{\partial H_0}{\partial d_n} \gtrless 0 \quad \iff \quad & W(d_n\left[\frac{1}{(1-d_n)}\left[\frac{1}{(1-d_n)} - (1-s_{n+1,0})\right]\right] - \frac{1}{(1-d_n)^2}\left[H_n\right] \gtrless \\ & -W(1-d_n) + \frac{1}{(1-d_n)}\left[-s_{n+1,0}W(s_{n,0})\right]\end{aligned} \tag{13}$$

Note that the RHS of the condition is necessarily negative. Regarding the LHS of the condition, we have, since d_n is very low, that the LHS can be approximated by:

$$(s_{n+1,0})\, W(d_n) - \frac{1}{(1-d_n)^2}\left[H_n\right] \tag{14}$$

When mortality d_n is low and increasing with age, the amount of information revealed by the event of a death at an age n, i.e. $W(d_n)$, is much higher than the mathematical expectation of the amount of information revealed by the event of a death at an age $i \geq n$, i.e. H_n, so that the LHS is strictly positive, and thus exceeds the RHS, which is strictly negative. Hence we have $\frac{\partial H_0}{\partial d_n} > 0$. Thus, when considering young adults, the effect of a change of mortality risk on lifetime entropy is positive.

Combining this with what we know concerning childhood, we see that two cases can arise concerning the low threshold a_1:

- Either infant mortality is lower than $\bar{d}_0$, so that it is the case that, during childhood and young adulthood, we have $\frac{\partial H_0}{\partial d_n} > 0$, in which case the threshold age equals 0, and we have $a_1 = 0$;
- Or infant mortality is higher than $\bar{d}_0$, so that we first have that a higher mortality risk reduces entropy during early childhood, and, then, starts raising entropy during either late childhood or young adulthood, so that there must exist a threshold age $a_1 > 0$ below which $\frac{\partial H_0}{\partial d_n} < 0$ and above which $\frac{\partial H_0}{\partial d_n} > 0$.

Consider now the existence of second threshold age a_2.

We know from above that, during young adulthood, we have $\frac{\partial H_0}{\partial d_n} > 0$. If the mortality risk is, during adulthood, increasing monotonically with age, the impact of a rise of the risk of death on lifetime entropy at birth becomes smaller and smaller as the mortality risk goes up. To see this, rewrite the condition of Proposition 1 as:

$$\frac{\partial H_0}{\partial d_n} \gtrless 0 \iff W(d_n) \gtrless \frac{H_{n+1}}{s_{n+1,k}} - W(s_{n,k}) \tag{15}$$

Take now the second-order derivative with respect to d_n. Since $s_{n,k}$ and H_{n+1} do not depend on d_n, we have:

$$\frac{\partial^2 H_0}{\partial d_n^2} = \underbrace{W'(d_n)}_{-} + \underbrace{\frac{H_{n+1}(-s_{n,k})}{(s_{n+1,k})^2}}_{-} < 0 \tag{16}$$

Thus the impact of d_n on lifetime entropy is positive but decreasing. Thus, as mortality risk rises with age (beyond age 20), the rise in lifetime entropy becomes smaller and smaller.

Does there exist a level of d_n such that $\frac{\partial H_0}{\partial d_n} = 0$ and then turns to be negative? The answer is affirmative. To see this, take very high ages. As n tends to be very large, we have:

$$\frac{\partial H_0}{\partial d_n} = W(s_{n,0}) + W(d_n) - \frac{H_{n+1}}{s_{n+1,0}} = \frac{s_{n+1,0}W(s_{n,0}) - H_{n+1}}{s_{n+1,0}} + W(d_n) \tag{17}$$

When n is very large, we have that $s_{n,0}$ and $s_{n+1,0}$ tend to 0, so that the above expression can be approximated by:

$$\frac{\partial H_0}{\partial d_n} = \frac{\sim 0 - H_{n+1}}{\sim 0} + W(d_n) < 0 \tag{18}$$

Thus, at high ages, we have that $\frac{\partial H_0}{\partial d_n} < 0$.Thus, since we have $\frac{\partial H_0}{\partial d_n} > 0$ during young adulthood, there must exist a second threshold age, a_2, below which $\frac{\partial H_0}{\partial d_n} > 0$ and above which $\frac{\partial H_0}{\partial d_n} < 0$.

This completes the proof of the existence of a threshold a_2. Note that the uniqueness of that second threshold age is guaranteed by the monotonicity of mortality with age for adults.

Finally, note that, when n is very large, and tends to the maximum age M, we have $H_{n+1} \to 0$, so that, using L'Hôpital's Rule, we have, from the above condition, that: $\frac{\partial H_0}{\partial d_n}$ tends towards 0. Thus, although the impact of a rise of d_n on H_0 is negative when $n > a_2$, this negative effect tends to vanish to zero when the age n approaches the maximum age M. ■

Proposition 2 states that, under general conditions on the age-mortality pattern, the effect of a variation of the probability of death d_n on lifetime entropy varies with age n. Proposition 2 states that there exists two threshold ages: first, a low threshold age a_1, below which a rise of d_n decreases lifetime entropy, and above which a rise of d_n increases lifetime entropy, at least until a second threshold age a_2 is reached, beyond which a rise of d_n decreases lifetime entropy. Thus, in the light of Proposition 2, it appears that how a variation of the probability of death affects lifetime entropy varies significantly with age: for very low ages and very high ages, a rise of d_n decreases lifetime entropy, whereas for intermediate ages a rise of d_n increases lifetime entropy.

When interpreting Proposition 2, one should be cautious about the level of the low threshold age a_1. Actually, as discussed in the proof, it is possible that, when infant mortality is sufficiently low, the first threshold age equals 0, so that, for all ages below a_2, a rise of d_n increases lifetime entropy.

4 Empirical illustration

In order to illustrate the existence of the two threshold ages around which the relation between mortality risk and lifetime entropy varies, this section uses life tables for France from the Human Mortality Database. For the simplicity of presentation, this section will use, for women and men, five period life tables, for years 1816, 1900, 1950, 1980 and 2016.

Figure 1 shows, for each of those life tables, the impact of a variation of the probability of death d_n on lifetime entropy at birth H_0 as a function of the age n (x axis). Figure 1 focuses on French women, but a quite similar picture prevails also for men (see in the Appendix).

Several observations can be made, which allow us to illustrate the results obtained in the previous sections.

A first important observation is that, based on the survival conditions prevailing in 1816, 1900 and 1950, there exists a strictly positive low threshold age a_1 below which a rise of d_n reduces lifetime entropy at birth. However, when considering the life tables for 1980 and 2016, that low threshold age has vanished to 0. That result is due to the strong fall of infant mortality in the second part of the 20th century.

A second observation concerns the existence of a high threshold age a_2, beyond which a rise of the probability of death reduces H_0. That second threshold age is shown to have moved significantly to the right, i.e. to higher values, when shifting from the 1816 life table to the 2016 life table.

Figure 1 thus shows that, during the major part of life, a higher probability of death at a given age tends to increase lifetime entropy at birth. However, the size of that age interval has tended to vary over time. That age interval was much shorter when considering life tables of 1816 and 1900, because, at those times, there was a low age interval, during childhood, in which a rise of d_n implied a fall of lifetime entropy, and, also, because, at those times, the second threshold age was much lower, and around age 60. Under contemporary survival conditions, the first threshold age has vanished to 0 and the second threshold age is much higher, at about 80 years.

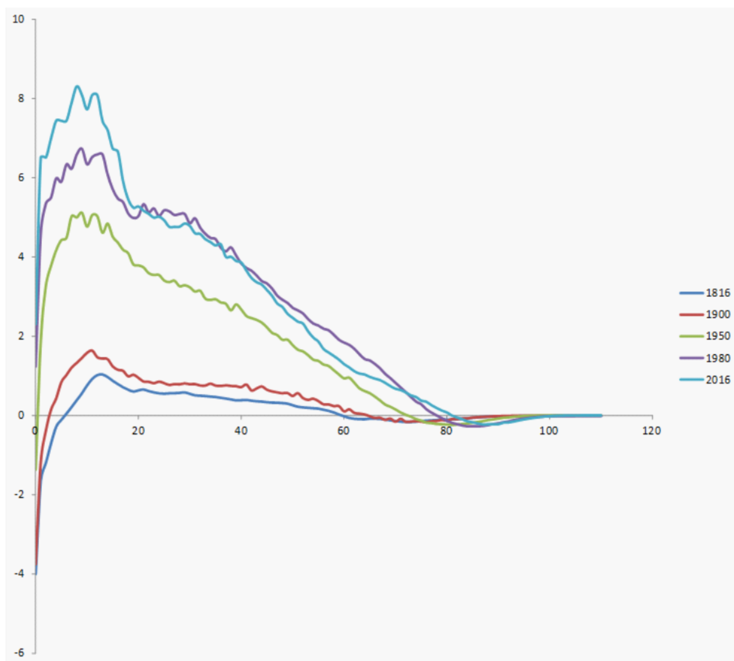

Figure 1: Effect of a variation of d_n on H_0 as a function of the age n, French women.

In order to have a more accurate view of those changes, Table 1 summarizes the levels of the two threshold ages a_1 and a_2 for men and women, under the five life tables under comparison. That table shows that the low threshold age has tended to decrease over time, and to vanish to zero. This change is due to the fall of infant mortality. As such, Table 1 illustrates well the above discussions in the proof of Proposition 2. Table 1 also shows that the second threshold age has tended to increase over time.

years	women		men	
	a_1	a_2	a_1	a_2
1816	6 years	60 years	6 years	60 years
1900	3 years	65 years	3 years	60 and 62 years
1950	1 year	73 years	1 year	68 years
1980	0 year	79 years	0 year	70 years
2016	0 year	81 years	0 year	81 years

Table 1: Threshold ages for women and men, France.

Another interesting observation concerning Table 1 is the lengthening, over time, of the age interval during which a rise of the probability of death tends to increase lifetime entropy. That age interval can be measured as the difference between the two threshold ages, i.e. $a_2 - a_1$. It was equal, for women, to 54 years in 1816, and is as large as 81 years for 2016. Thus the part of life during which a rise of mortality increases lifetime entropy is now much larger than it used to be in the past.

When considering Table 1, a quite surprising result consists of the existence of two high threshold ages for men in 1900. That result is due to the non-monotonicity of the probability of death d_n around ages 60-61 for French men in 1900. That non-monotonicity explains the non-uniqueness of the high threshold age. Note that this does not infirm the validity of Proposition 2, since Proposition 2 assumed the monotonicity of the mortality age-pattern during adulthood, which was a sufficient condition for the uniqueness of the high threshold age a_2. The 1900 life table does not satisfy that monotonicity condition.

5 Concluding remarks

This paper derived a necessary and sufficient condition under which a rise of the probability of death at a given age increases lifetime entropy measured by Shannon's entropy index defined to the base 2. That condition is actually quite intuitive: a rise in the probability of death at age n (conditionally on survival to age n) increases lifetime entropy at age $k \leq n$ if and only if the quantity of information revealed by the event of a death at age n, measured, by means of Wiener's entropy, by $W(s_{n,k}) + W(d_n)$, exceeds lifetime entropy at age $n+1$ divided by the probability to survive from age k to age $n+1$.

That condition can be used to show analytically the existence of two threshold ages a_1 and a_2. When the age is below a_1, a rise of the risk of death decreases lifetime entropy at birth, whereas when the age lies between a_1 and a_2, it increases lifetime entropy at birth. Then, for ages exceeding a_2, a rise in the probability of death reduces lifetime entropy.

Using French life tables over the last two centuries, we showed that the first threshold age a_1 has tended to decrease over time, from about 6 years in 1816 to 0 year in 1980 and after. On the contrary, the threshold age a_2 has tended to increase, from about 60 years in 1816 to 81 years in 2016. As a consequence of those two patterns, an increasingly larger period of life is characterized by a positive relation between the risk of death and lifetime entropy in comparison to more distant epochs.

All in all, this study shows that the relation between the risk of death and lifetime entropy varies with age, and that this relation has also changed over the last two centuries. The existence of a *unique* strictly positive threshold age, as studied in Zhang and Vaupel (2009) and Aburto et al (2019), dates back to the second part of the 20th century. Before that, there used to be another, lower, threshold age, below which a rise of mortality risk leads to a fall of lifetime entropy.

To conclude, it should be stressed that those findings are most relevant for lifecycle analysis, especially for the understanding of individual reactions to mortality shocks due, for instance, to epidemics, pollution peaks or heat waves. Since decisions in terms of insurance or precautionary saving depend not only on life expectancy, but, also, on the risk about the duration of life, it is important to quantify how mortality shocks affect the risk about the duration of life. The present paper reveals that the sign of the effect of a mortality shock on lifetime entropy varies with the age, and identifies some critical age thresholds at which the sign of the effect of a mortality shock on lifetime entropy changes. As such, this paper casts original light on a key effect of mortality shocks, and, hence, on a channel through which those shocks may affect individual behaviors and aggregate economic outcomes.

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7 Appendix

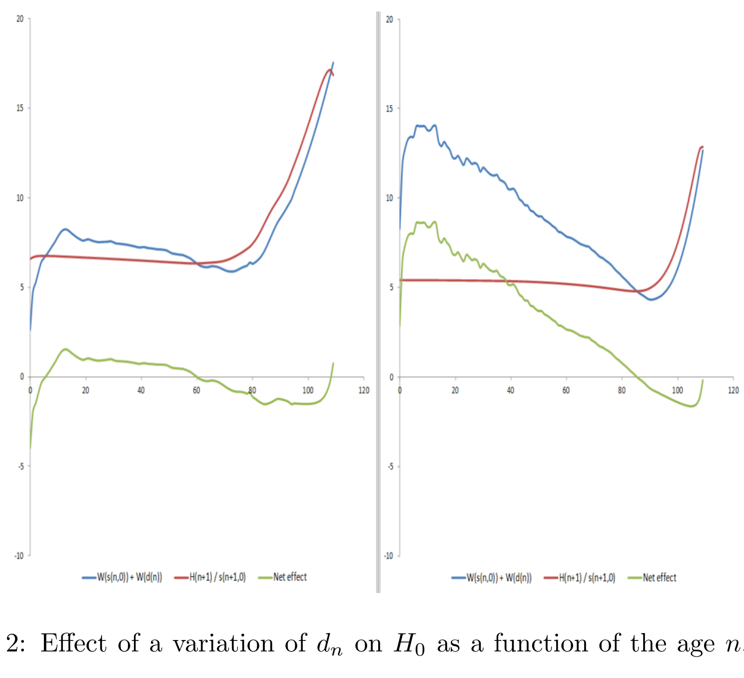

Figure 2: Effect of a variation of d_n on H_0 as a function of the age n, French men.