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ABSTRACT. The clusters considered in this paper are seen as morphisms between small arbitrary diagrams in a given locally small category $\mathscr{C}$. They have initially been introduced to extend to all small diagrams the results for filtered diagrams, by exhibiting a very basic presentation of the formula used in the definition of the category $\mathrm{Ind}(\mathscr{C})$ of ind-objects in $\mathscr{C}$. They constitute a category $\mathrm{Clu}\,(\mathscr{C})$ which contains $\mathrm{Ind}(\mathscr{C})$. We study these clusters, their construction and composition. Thus we provide any user with the means to generate clusters and perform calculations with them. So we can give a simple proof of the fact that $\mathrm{Clu}\,(\mathscr{C})$ is a strict free cocompletion of $\mathscr{C}$ for all small diagrams, determined up to isomorphism. We compare it to some other cocompletion problems.

1. Introduction

The purpose of this article is to consider the *strict free cocompletion problem* (as precisely introduced here in Definition 4.1, making a clear distinction between the strict case and the loose case), and its specific solution (ESFC) i.e. Theorem 4.4 [A. Ehresmann, 1981]), constructed with the notion of *cluster* (Definition 3.1). We will give details for the construction and proof, and its relationship to some other cocompletion processes.

In the preliminary Section 2, standard material is provided as a language to carry out the constructions in a way that emphasizes the role of the "connected component functor" π_0 in relation with the so called *flows* (Definition 2.2). For this purpose, we introduce the discrete cofibration $\Xi_{P,Q}$ universally associated with two diagrams P and Q. Thus it would be useful to characterize the notion of cluster by starting from the basic datum π_0 only, because this prepares future works where the functor π_0 will be replaced by another functor.

In Section 3, we introduce the central notion in this article, that of a *cluster*, in Definition 3.1. We indicate how to generate clusters from flows (Lemma 3.6), which is necessary to compose clusters, or if we need to know whether or not there is a cluster between two diagrams P and Q. Moreover, Lemma 3.6 may be useful in the application

This article is dedicated to our great friend Bob Rosebrugh, in honor of his very significant contributions to category theory, and in thanks for his remarkable work as managing editor of the journal TAC, and moderator of the Category Theory mailing list for a long time.

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of clusters to the representation of "emergence" in evolutive systems, as introduced in [Ehresmann & Vanbremeersch, 1987], [Ehresmann & Vanbremeersch, 2007].

Thanks to the notion of *precluster* (Definition 3.1), we construct the elementary presentation of the category $\mathrm{Clu}(\mathscr{C})$, with its simple relation to the category of flows. We introduce two actions, namely *parallelization* in Lemma 3.7 and *binding* in Lemma 3.10, which will be useful in Section 4 to establish Theorem 4.4.

Then, we show that clusters satisfy the Lim-Colim formula, and so are related to some natural transformations. This shows that the category $\mathrm{Clu}(\mathscr{C})$ is isomorphic to an extension of Grothendieck's construction of $\mathrm{Ind}(\mathscr{C})$, the category of ind-objects in $\mathscr{C}$, to non-necessarily filtered diagrams. Via this extension, denoted $\mathrm{LClu}(\mathscr{C})$ in Proposition 3.13, $\mathrm{Clu}(\mathscr{C})$ "contains" $\mathrm{Ind}(\mathscr{C})$.

In Section 4, we propose a complete and elementary proof of Theorem 4.4. From this theorem, we can deduce Grothendieck's result on $\mathrm{Ind}(\mathscr{C})$.

In the concluding Section 5, we compare the construction of $\mathrm{Clu}(\mathscr{C})$ with some other cocompletions. In short, among the several ways to cocomplete categories, strictly or loosely, the cocompletion of $\mathscr{C}$ that yields $\mathrm{Clu}(\mathscr{C})$ seems to be one of the most efficient, for instance for explicit constructions of colimits.

2. Notation: comma categories, flows, connected components

Throughout this article, the category of sets is **Set**, the category of small categories is **Cat**, $\mathscr{C}$ is a fixed locally small category, $\mathscr{P}$, $\mathscr{Q}$, ..., are always small categories and $P : \mathscr{P} \to \mathscr{C}$, $Q : \mathscr{Q} \to \mathscr{C}$, ..., are always small diagrams.

In this preliminary Section 2, we recall the definition of a comma category $(P \downarrow Q)$, and we introduce the notion of flow, the functor $K_{P,Q}$ which exhibits Q-(co)shapes for objects $P(p)$; then come into play connected components, the functors π_0 and $\mathrm{CC}_{P,Q}$, and the associated split cofibration $\Xi_{P,Q}$, which informally express the action of π_0 on Q-(co)shapes for objects $P(p)$. It should also be noted that for explicit calculations, the notation $[g]_p$ given in 2.8 will then be very convenient and frequently used.

2.1. DEFINITION. [The comma category $(P \downarrow Q)$] *[Lawvere, 1963] The* comma category *of two small diagrams $P : \mathscr{P} \to \mathscr{C}$, $Q : \mathscr{Q} \to \mathscr{C}$ in a locally small category $\mathscr{C}$, is the small category denoted by $(P \downarrow Q)$ defined as follows. Its objects are the triples (p, g, q) — also denoted here $g : p \rightharpoonup q$ — such that $p \in \mathrm{Ob}(\mathscr{P})$, $q \in \mathrm{Ob}(\mathscr{Q})$, and $g : P(p) \longrightarrow Q(q) \in \mathrm{Mor}(\mathscr{C})$. A morphism $(p, g, q) \longrightarrow (p', g', q')$ is a pair (a, b), where $a : p \longrightarrow p' \in \mathscr{P}$ and $b : q \longrightarrow q' \in \mathscr{Q}$ make the following square commute:*

$$\begin{array}{ccc} P(p) & \xrightarrow{P(a)} & P(p') \\ {\scriptstyle g}\downarrow & = & \downarrow{\scriptstyle g'} \\ Q(q) & \xrightarrow[Q(b)]{} & Q(q') \end{array}$$

We also consider the "source" functor $S_{P,Q}$ with values on objects and arrows given by

$$S_{P,Q} : (P \downarrow Q) \to \mathscr{P} : (p, g, q) \mapsto S_{P,Q}(p, g, q) = p, \ (a, b) \mapsto S_{P,Q}(a, b) = a.$$

2.2. DEFINITION. [Ord-category of flows] *A* flow $G : P \to Q$ *is the datum of a set G of objects of $(P \downarrow Q)$. Given two flows $G : P \to Q$ and $H : Q \to R$, their composition is*

$$H \circ G = \{m : p \rightharpoonup r \mid \exists q \in \mathrm{Ob}\,(\mathscr{Q})\,, \exists g : p \rightharpoonup q \in G, \exists h : q \rightharpoonup r \in H\,(m = hg)\}\,.$$

We denote by $\mathbb{F}\mathrm{low}(\mathscr{C})$ the category enriched in ordered sets which, as a 2-category, has as objects small diagrams in $\mathscr{C}$, as 1-morphisms flows with the composition law $\circ$, and as 2-morphisms from $G : P \to Q$ to $G' : P \to Q$ inclusions $G \subset G'$. Its category of 1-morphisms is denoted by $\mathrm{Flow}\,(\mathscr{C})$*, and the set $\mathrm{Hom}_{\mathbb{F}\mathrm{low}(\mathscr{C})}(P, Q)$ is simply denoted by* $\mathrm{Flow}\,(\mathscr{C})\,(P, Q)$*.*

2.3. REMARK. For any object C in $\mathscr{C}$, we consider the diagram

$$u_C^{\mathscr{C}} : \{C\} \longrightarrow \mathscr{C} : C \mapsto u_C^{\mathscr{C}}(C) = C$$

and the comma category $\left(u_C^{\mathscr{C}} \downarrow Q\right)$ is shortly denoted $(C \downarrow Q)$. For $f : C \to C'$, we have the flow $u_f^{\mathscr{C}} = \{f : C \rightharpoonup C'\}$. So an arrow f in $\mathscr{C}$ "is" a flow, namely $u_f^{\mathscr{C}}$. Hence, as a set of morphisms, a flow is trivially a generalisation of a morphism, and we have an embedding

$$u^{\mathscr{C}} : \mathscr{C} \hookrightarrow \mathrm{Flow}\,(\mathscr{C})\,.$$

2.4. EXAMPLE. [Binary relations as flows]

1. If we restrict our diagrams to discrete diagrams, i.e. families of objects in $\mathscr{C}$, $P : I \to \mathscr{C}$, and if, moreover, we consider the case when $\mathscr{C} = \mathbb{1}$, then, thus restricted, $\mathrm{Flow}\,(\mathscr{C})$ just becomes $\mathrm{Rel}(\mathbf{Set})$, the category of binary relations between sets.

2. Let us consider the case $\mathscr{C} = \mathbf{Set}$. Given two sets E and F, we consider the diagrams $\alpha_E : E \to \mathbf{Set} : x \mapsto \{x\}$ and $\alpha_F : F \to \mathbf{Set} : x \mapsto \{x\}$. Then a binary relation R from E to F is exactly a flow $\alpha(R)$ from α_E to α_F, and so we have an embedding
$$\alpha : \mathrm{Rel}(\mathbf{Set}) \hookrightarrow \mathrm{Flow}(\mathbf{Set}).$$

2.5. DEFINITION. [The functor $K_{P,Q}$] *For each object $p \in \mathscr{P}$, we consider the comma category $\left(P_{|p} \downarrow Q\right)$ where the diagram $P_{|p} : \{p\} \longrightarrow \mathscr{C}$ is defined by setting $P_{|p}(p) = P(p)$. It is the subcategory of $(P \downarrow Q)$ whose objects are the $g : p \rightharpoonup q$ or triples of the form (p, g, q) and morphisms are pairs (id_p, b) with b a morphism in $\mathscr{Q}$. We introduce also the dual*

$$K_{P,Q}(p) = \left(P_{|p} \downarrow Q\right)^{\mathrm{op}},$$

and this determines a functor $K_{P,Q} : \mathscr{P}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$, with, for $a : p' \to p$, $K_{P,Q}(a)(g) = gP(a)$.

2.6. REMARK. With the notion of cofibration as in [Grothendieck, 1971], we recover the comma category from this $K_{P,Q}$ by taking the split cofibration $q_{K_{P,Q}} : \int K_{P,Q} \longrightarrow \mathscr{P}^{\mathrm{op}}$ associated to $K_{P,Q}$, which is the dual of the source projection $S_{P,Q}$ (see Definition 2.1).

2.7. DEFINITION. [The connected component functor π_0] *Let $\mathscr{X}$ be a small category. We define an equivalence relation $\sim_{\mathscr{X}}$ on its set of objects by $X \sim_{\mathscr{X}} X'$ iff there is a finite "zigzag" of morphisms, in "alternate" directions, connecting X to X':*

$$X \longrightarrow X_1 \longleftarrow X_2 \longrightarrow \cdots \longleftarrow X_{n-2} \longrightarrow X_{n-1} \longleftarrow X'.$$

The equivalence class of X for this relation will be denoted by $[X]_{\mathscr{X}}$ and named the connected component of X. *The* connected component functor $\pi_0 : \mathbf{Cat} \to \mathbf{Set}$ *is given by*

$$\pi_0(\mathscr{X}) = \{[X]_{\mathscr{X}} \mid X \subset \mathrm{Ob}(\mathscr{X})\},\ \pi_0(F : \mathscr{X} \to \mathscr{X}') : [X]_{\mathscr{X}} \mapsto [F(X)]_{\mathscr{X}'}.$$

Given a set E, let $\mathrm{Dis}(E)$ be the discrete category with objects the elements of E and morphisms the identities on these objects. Then, it is well known that the construction Dis is left adjoint to Ob, and π_0 is left adjoint to Dis:

$$\mathrm{Hom}_{\mathbf{Cat}}(\mathrm{Dis}(E), \mathscr{X}) \simeq \mathrm{Hom}_{\mathbf{Set}}(E, \mathrm{Ob}(\mathscr{X})),$$

$$\mathrm{Hom}_{\mathbf{Set}}(\pi_0(\mathscr{X}), E) \simeq \mathrm{Hom}_{\mathbf{Cat}}(\mathscr{X}, \mathrm{Dis}(E)).$$

The unit of the second adjunction is the quotient map

$$\kappa_{\mathscr{X}} : \mathscr{X} \longrightarrow \mathrm{Dis}(\pi_0(\mathscr{X})) : X \mapsto \kappa_{\mathscr{X}} = [X]_{\mathscr{X}}.$$

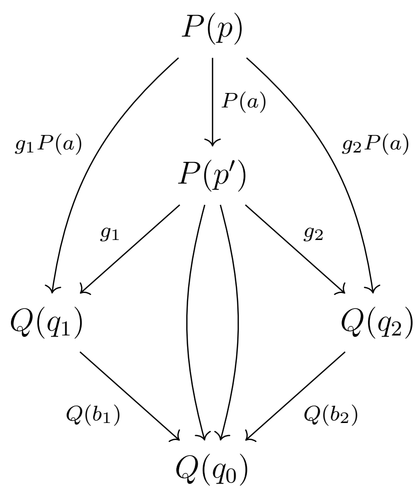
Thence this map is the unit of a monad $\mathrm{Dis} \circ \pi_0$ on $\mathbf{Cat}$ of which algebras are sets.

2.8. NOTATION. $[[g]_p]$ *The connected component of $g : p \rightharpoonup q$ in $\left(P_{|p} \downarrow Q\right)$, that is to say $\kappa_{\left(P_{|p} \downarrow Q\right)}(g : p \rightharpoonup q) = [g]_{\left(P_{|p} \downarrow Q\right)}$, will be shortly denoted by $[g]_p$, without necessarily specifying P and Q when there is no risk of confusion from the context.*

2.9. REMARK. Two objects $g : p \rightharpoonup q$ and $g' : p \rightharpoonup q'$ in $\left(P_{|p} \downarrow Q\right)$ that are connected in $(P \downarrow Q)$ may or may not be connected in $\left(P_{|p} \downarrow Q\right)$, because the zigzags in $\left(P_{|p} \downarrow Q\right)$ must have id_p as components in $\mathscr{P}$. This is illustrated by the following easy example, which intuitively underpins the figure in Lemma 3.6 and is used in the proof of Lemma 3.7.

With sources $\mathscr{P} = \{p \xrightarrow{a} p'\}$, and $\mathscr{Q} = \{q_1 \xrightarrow{b_1} q_0 \xleftarrow{b_2} q_2\}$, we consider two diagrams $P : \mathscr{P} \to \mathscr{C}$ and $Q : \mathscr{Q} \to \mathscr{C}$ in a category $\mathscr{C}$, as below, where we suppose that the two arrows $Q(b_1)g_1$ and $Q(b_2)g_2$ are different, but that

$$Q(b_1)g_1P(a) = Q(b_2)g_2P(a).$$



More precisely, in this case, we have two different connected components of $\left(P_{|p'} \downarrow Q\right)$ that are included in the same connected component of $(P \downarrow Q)$. Indeed, $\left(P_{|p} \downarrow Q\right)$ is connected, consisting of the shorter zigzag:

$$g_1 P(a) \to Q(b_1) g_1 P(a) = Q(b_2) g_2 (P(a) \leftarrow g_2 P(a),$$

but $\left(P_{|p'} \downarrow Q\right)$ is not connected, consisting of two isolated arrows, extracted from $(P \downarrow Q)$:

$$Q(b_1) g_1 \leftarrow g_1 \text{ and } g_2 \to Q(b_2) g_2.$$

The two previous components are welded and the comma category $(P \downarrow Q)$ is connected. It is actually a zigzag of 6 arrows between 7 objects :

$$Q(b_1) g_1 \leftarrow g_1 \leftarrow g_1 P(a) \to Q(b_1) g_1 P(a) = Q(b_2) g_2 P(a) \leftarrow g_2 P(a) \to g_2 \to Q(b_2) g_2.$$

We conclude this section by introducing two functors that will be instrumental to make obvious the Lim-Colim formula in Proposition 3.11.

2.10. DEFINITION. [The functor $\mathrm{CC}_{P,Q}$ and the associated split cofibration $\Xi_{P,Q}$] *We introduce the functor*

$$\mathrm{CC}_{P,Q} = \pi_0 \circ K_{P,Q} : \mathscr{P}^{op} \to \mathbf{Cat} : \begin{cases} p & \mapsto \quad \mathrm{CC}_{P,Q}(p) = \pi_0\left(P_{|p} \downarrow Q\right) \\ a : p' \longrightarrow p & \mapsto \quad \begin{cases} \mathrm{CC}_{P,Q}(p) & \longrightarrow \quad \mathrm{CC}_{P,Q}(p') \\ [g]_p & \longmapsto \quad [gP(a)]_{p'} \end{cases} \end{cases}$$

that associates to $p \in \mathscr{P}$ *the set of "connected components at* p*" (CC) in* $\left(P_{|p} \downarrow Q\right)$*.*

The split cofibration associated to $\mathrm{CC}_{P,Q}$ *(see [Beurier, 2020]) is denoted by*

$$\Xi_{P,Q} : ((P \downarrow Q)^{\mathrm{op}})^{\mathrm{coarse}} \longrightarrow \mathscr{P}^{\mathrm{op}}.$$

Its source is the "coarsened" category $((P \downarrow Q)^{\mathrm{op}})^{\mathrm{coarse}}$ of P and Q, *where an object is a labelled connected component* $\left(p, [g]_p\right)$, $[g]_p \in \pi_0(P_{|p} \downarrow Q)$, $p \in \mathscr{P}$*; and an arrow*

$\left(p, [g]_p\right) \longrightarrow \left(p', [g']_{p'}\right)$ *is a pair* $\left(a, [g]_p\right)$, $a : p' \longrightarrow p \in \mathscr{P}$ *such that* $[g']_{p'} = [gP(a)]_{p'}$. *The composition of consecutive arrows* $\left(a, [g]_p\right) : [g]_p \longrightarrow [g']_{p'}$ *and* $\left(a', [g']_{p'}\right) : [g']_{p'} \longrightarrow [g'']_{p''}$ *is* $\left(a'a, [g]_p\right) : [g]_p \longrightarrow [g'']_{p''}$. *And given an arrow* $\left(a, [g]_p\right)$ *we have* $\Xi_{P,Q}\left(a, [g]_p\right) = a$. *This* $\Xi_{P,Q}$ *is the free discrete cofibration generated by* $S^{\mathrm{op}}_{P,Q}$ *(see Definition 2.1, Remark 2.6).*

3. From the category of flows to the category of clusters, via preclusters

From now on, we will add conditions of naturality and connectedness to flows: this is necessary to properly describe cocompletions. We use the notations of Section 2 and in particular those of Notation 2.8.

We construct the category of clusters $\mathrm{Clu}(\mathscr{C})$ via the operation $(-)^\sigma$ introduced below in Lemma 3.6, as an extension of a subcategory $\mathrm{PreClu}(\mathscr{C})$ of $\mathrm{Flow}(\mathscr{C})$, the obvious category of flows (Definition 2.2).

So we start with the main definition in this article (Definition 3.1), with the rather technical notion of *precluster* and the effective notion of *cluster*. With the operator $(-)^\sigma$ (Proposition 3.5) we can finish the elementary description of $\mathrm{Clu}(\mathscr{C})$, its elements and its composition, that is to say, Theorem 3.9.

In this movement, we isolate two lemmas that will be useful later: "parallelization" 3.7 and "binding" 3.10. In Example 3.3, we also consider the case of a cluster generated by a morphism of diagrams, which will be useful in Section 5.

Afterwards, we show that clusters admit a Lim-Colim formula (Proposition 3.11), we establish the link between $\mathrm{Clu}(\mathscr{C})$ and the Grothendieck construction with presheaves by exhibiting the category $\mathrm{LClu}(\mathscr{C})$.

3.1. DEFINITION. [Flow, preclusters and clusters] *[A. Ehresmann, 1981], [Ehresmann & Vanbremeersch, 1987] Let P and Q be two small diagrams in a locally small category $\mathscr{C}$. With a* flow $G \subset \mathrm{Ob}(P \downarrow Q)$ *(cf. Definition 2.2), we define $G(p)$ as the subset of G consisting of all the arrows in G starting with label p, that is to say:*

$$G(p) = \{g : p \rightharpoonup q \mid g \in G\} = G \cap \mathrm{Ob}\left(P_{|p} \downarrow Q\right), \quad G = \biguplus_{p \in \mathrm{Ob}(\mathscr{P})} G(p).$$

A precluster *from P to Q is a flow G that verifies the conditions (CLU-1), (CLU-2), (CLU-3) given below:*

- *(CLU-1) for all $p \in \mathscr{P}$, there exists $g : p \rightharpoonup q \in G$ for some $q \in \mathscr{Q}$ i.e. $G(p) \neq \emptyset$.*
- *(CLU-2) for all $p \in \mathscr{P}$, if $g : p \rightharpoonup q \in G$ and $g' : p \rightharpoonup q' \in G$, then they are connected in the comma category $\left(P_{|p} \downarrow Q\right)$, i.e. $[g]_p = [g']_p$*
- *(CLU-3) if $a : p' \longrightarrow p \in \mathscr{P}$ and $g : p \rightharpoonup q \in G(p)$, then $gP(a) : p' \rightharpoonup q \in G(p')$, i.e. $\{gP(a) \mid g \in G, a : p' \longrightarrow p\} \subset G(p')$*

A cluster *from P to Q is a maximal precluster, that is, a precluster verifying (CLU-4):*

(CLU-4) G is maximal for inclusion $\subseteq$ among the preclusters from P to Q

The set of preclusters from P to Q is denoted by $\mathrm{PreClu}(\mathscr{C})(P,Q)$, and the set of clusters from P to Q is denoted by $\mathrm{Clu}\,(\mathscr{C})\,(P,Q)$.

3.2. EXAMPLE. (Functions) Given two maps between sets (seen as discrete categories), $f : X \to E$ and $g : Y \to E$, then a cluster from f to g over E is exactly a map $m : X \to Y$ with $f = g \circ m$.

3.3. EXAMPLE. (Arrows, cocones, morphisms of diagrams) An arrow f in a category "is" a cluster, when seen as $u_f^{\mathscr{C}}$ in the definition of flows.

More generally, with P a diagram in $\mathscr{C}$, and with $\alpha = (\alpha_p : P(p) \longrightarrow A)_{p\in\mathrm{Ob}(\mathscr{P})}$ a family of morphisms in $\mathscr{C}$, we introduce a flow $\widehat{\alpha} = \{(p, \alpha_p, A) \mid p \in \mathscr{P}\}$. Then $\widehat{\alpha}(p) = \{(p, \alpha_p, A)\} = [\alpha_p]_p$ and α is a cocone if and only if $\widehat{\alpha}$ is a cluster $\widehat{\alpha} : P \longrightarrow u_A^{\mathscr{C}}$. This cluster $\widehat{\alpha}$ is said to be *generated by* the cocone α.

Given P and Q two diagrams, a *morphism of diagrams* from P to Q is a pair (F, f) with $F : \mathscr{P} \to \mathscr{Q}$ and a natural transformation $f : P \Rightarrow Q \circ F$. To such a datum we associate a cluster $[F, f]$ from P to Q with $[F, f](p) = [f_p]_p$.

Our next Proposition 3.4 explains how to identify our basic elementary notion of clusters (Definition 3.1) with sections of $\Xi_{P,Q}$, which will make the Lim-Colim formula obvious (Proposition 3.11).

3.4. PROPOSITION. [Clusters as "pw-atomic" subfunctors of $\mathrm{CC}_{P,Q}$ or sections of $\Xi_{P,Q}$] *A flow $G \subset \mathrm{Ob}\,(P \downarrow Q)$ is a cluster if and only if G verifies (CLU-3) and each $G(p)$ is a connected component of $\left(P_{|p} \downarrow Q\right)$, i.e. $G(p) \in \mathrm{CC}_{P,Q}(p)$.*

In other words, it is a datum $G(-)$ such that $p \mapsto \{G(p)\}$ is on $\mathscr{P}^{\mathrm{op}}$ a subfunctor of $p \mapsto \mathrm{CC}_{P,Q}(p)$ (see Definition 2.10). Hence its value for each $a : p' \longrightarrow p$ is $([g]_p \mapsto [gP(a)]_{p'})$. Such a sub-functor will be called "pw-atomic" (pw = pointwise) since its values, which are parts of $\mathrm{CC}_{P,Q}(p)$, are atoms (singletons). Of course it is a datum equivalent to a section of $\Xi_{P,Q}$ (cf. Definition 2.10).

PROOF. A detailed proof is given in [Beurier, 2020, Proposition 4.4.4., page 121]. ■

3.5. PROPOSITION. [Saturation of flows] *Let $G : P \longrightarrow Q$ be any flow. We define :*

$$G^{\Sigma}(p) = \{[gP(a)]_p \mid p' \in \mathrm{Ob}\,(\mathscr{P})\,,\, a : p \to p',\, (g : p' \rightharpoonup q) \in G(p'),\, q \in \mathrm{Ob}\,(\mathscr{Q})\}\,,$$

We denote by G^{σ} the saturation *of G, defined by:*

$$G^{\sigma}(p) = \bigcup_{C\in G^{\Sigma}(p)} C.$$

Such a G^{Σ} is a subfunctor of $\mathrm{CC}_{P,Q}$, and conversely any subfunctor S of $\mathrm{CC}_{P,Q}$ is of the form $S = G^{\Sigma}$ for $G(p) = \bigcup_{C\in S(p)} C$.

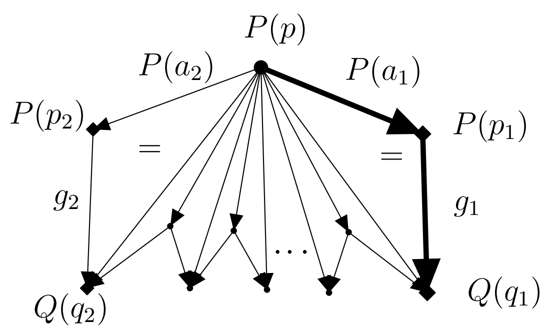
The consideration of arbitrary subfunctors of $CC_{P,Q}$ — instead of only pw-atomic ones as in Proposition 3.4 — is a convenient way to generate clusters from some flows G, by "saturation" G^σ satisfying a functoriality condition (at the level of connected components) (Proposition 3.5), as explained without its straightforward proof in the next Lemma 3.6. This lemma is the natural tool to exhibit all examples of clusters and see some flows G as *presentations of clusters* as their saturation G^σ. The composition in the category of clusters is achieved by simply saturating preclusters.

Item 4 of Lemma 3.6 formally expresses that a given cluster is completely defined by any "sub-flow" picking out one element of each connected component. Thus (CLU-3) is satisfied "up to connected components". Such a choice will be useful for instance to describe the functor $L_{\mathscr{C}}$ in Proposition 3.12.

3.6. LEMMA. [Cluster generation] *Given any flow $G : P \longrightarrow Q$ and G^σ as in Proposition 3.5 we have:*

1. *G^σ is a cluster if and only if G^Σ is a pw-atomic subfunctor of $\mathrm{CC}_{P,Q}$ (see Proposition 3.4). This condition is equivalent to saying that for all spans $p_1 \overset{a_1}{\leftarrow} p \overset{a_2}{\rightarrow} p_2$ in $\mathscr{P}$, and for all $g_1 : p_1 \rightharpoonup q_1$ and $g_2 : p_2 \rightharpoonup q_2$ in G, we have the functoriality condition:*

$$[g_1 P(a_1)]_p = [g_2 P(a_2)]_p$$



2. *For $G = (g_p)_p \in \prod_{p \in \mathrm{Ob}(\mathscr{P})} \mathrm{Ob}\left(P_{|p} \downarrow Q\right)$, G^σ is a cluster iff it verifies the functoriality condition of Item 1, that is to say for $g_{p_1} : p_1 \rightharpoonup q_{p_1}$ and $g_{p_2} : p_2 \rightharpoonup q_{p_2}$ in G*

$$[g_{p_1} P(a_1)]_{p_1} = [g_{p_2} P(a_2)]_{p_2}$$

3. *If G is a precluster, then G^σ is the unique cluster containing it, and for any $g \in G(p)$ we have $G^\sigma(p) = [g]_p$.*

4. *If G is a cluster, then for all $\Gamma \in \prod_{p \in \mathrm{Ob}(\mathscr{P})} G(p)$, $\Gamma^\sigma = G$.*

3.7. LEMMA. *(Parallelization by elongation) Let G be a flow from P to Q. We consider $g : p \rightharpoonup q$ and $g' : p' \rightharpoonup q'$ in G. If G is a precluster, and if p and p' are connected in $\mathscr{P}$, then g and g' are connected in $(P \downarrow Q)$. But conversely this last property does not imply that G is a precluster.*

PROOF. Let $(p_i)_{i \in I}$ be the objects of a zigzag connecting p' and p in $\mathscr{P}$. By (CLU-1), we associate a $g_i \in G(p_i)$ to each p_i, $g_i : P(p_i) \to Q(q_i)$. For each arrow $a : p_i \longrightarrow p_j$ of the zigzag, $g_j P(a) \in G(p_i)$ by (CLU-3). Since $g_i \in G(p_i)$ as well, it follows from (CLU2) that g_i and $g_j P(a)$ are connected in $\left(P_{|p_i} \downarrow Q\right)$, and notably we get a zigzag from q_i to q_j. By adding a convenient number of identities on p_i we get a zigzag from p_i to p_j with the same length as the zigzag from q_i to q_j, and consequently connecting in this way each zigzag in $\left(P_{|p_i} \downarrow Q\right)$ by means of the arrows of the initial zigzag in $\mathscr{P}$, we obtain a new zigzag in $(P \downarrow Q)$, called *parallelization* of the initial one in $\mathscr{P}$.

The converse is false because — see Remark 2.9 — connectedness in $(P \downarrow Q)$ does not imply connectedness in $(P_{|p} \downarrow Q)$ and thus (CLU-2). ∎

3.8. PROPOSITION. (Category of preclusters) *Given two preclusters $G : P \to Q$ and $H : Q \to R$, we consider their composition $H \circ G$ as flows over $\mathscr{C}$, that is to say, as in Proposition 2.2:*

$$(H \circ G)(p) = \bigcup_{q \in \mathrm{Ob}(\mathscr{Q})} \{hg : p \rightharpoonup r \mid g : p \rightharpoonup q, h : q \rightharpoonup r, g \in G, h \in H\}.$$

Then, $H \circ G$ is itself a precluster. Consequently, we get the category $\mathrm{PreClu}(\mathscr{C})$ of preclusters over $\mathscr{C}$, as a subcategory of $\mathrm{Flow}(\mathscr{C})$, hence an inclusion functor:

$$\mathrm{inc} : \mathrm{PreClu}(\mathscr{C}) \hookrightarrow \mathrm{Flow}(\mathscr{C}).$$

PROOF. Clearly $H \circ G$ satisfies (CLU-1) and (CLU-3), as G and H do. For (CLU-2), we have a connection from $g : p \rightharpoonup q$ to $g' : p \rightharpoonup q'$ in $\left(P_{|p} \downarrow Q\right)$, hence a connection from q to q' in $\mathscr{Q}$; with this last one, and the datum h, h', we apply the parallelization Lemma 3.7 to obtain a connection in $(Q \downarrow R)$ from h to h', which can be composed with the first connection between g and g', in order to connect hg to $h'g'$. ∎

Now, using Proposition 3.5 and Lemma 3.6 for the saturation operation G^σ, especially in the case of preclusters, we have the following theorem.

3.9. THEOREM. [Category of clusters] *We use Propositions 3.5 and 3.8.*

1. *We define the composition of clusters by*

$$H \odot G = (H \circ G)^\sigma.$$

With $G(p) = [g]_p$ and $H(q) = [h]_q$, $g : p \rightharpoonup q$, $h : q \rightharpoonup r$, we obtain, with $hg : p \rightharpoonup r$:

$$(H \odot G)(p) = [h]_q \odot [g]_p = [hg]_p.$$

This composition is associative and we get the category $\mathrm{Clu}(\mathscr{C})$ of clusters over $\mathscr{C}$.

2. *We have an embedding*

$$I_{\mathscr{C}} : \mathscr{C} \to \mathrm{Clu}(\mathscr{C}).$$

3. *We have an inclusion of sets* $\mathrm{Clu}(\mathscr{C}) \subset \mathrm{Flow}(\mathscr{C})$, *but, unlike* $\mathrm{PreClu}(\mathscr{C})$, $\mathrm{Clu}(\mathscr{C})$ is not *a subcategory of* $\mathrm{Flow}(\mathscr{C})$.

4. *For preclusters we have*

$$H^\sigma \odot G^\sigma = (H \circ G)^\sigma,$$

and so the operation $(-)^\sigma$ *determines a canonical functor*

$$(-)^\sigma : \mathrm{PreClu}(\mathscr{C}) \to \mathrm{Clu}(\mathscr{C}),$$

and the relationship between flows and clusters via preclusters is visualised by the span

$$\mathrm{Flow}(\mathscr{C}) \overset{\mathrm{inc}}{\hookleftarrow} \mathrm{PreClu}(\mathscr{C}) \overset{(-)^\sigma}{\to} \mathrm{Clu}(\mathscr{C}).$$

PROOF. Considering three clusters $G : P \to Q$, $H : Q \to R$ and $L : R \to S$, we have $((L \circ H)^\sigma \circ G)^\sigma = (L \circ (H \circ G)^\sigma)^\sigma$. This formula comes from the unicity in Item 3 of Lemma 3.6. Explicitly, we have

$$L \circ (H \circ G) \subset L \circ (H \circ G)^\sigma \subset (L \circ (H \circ G)^\sigma)^\sigma = L \odot (H \odot G),$$

and similarly $(L \circ H) \circ G \subset (L \odot H) \odot G$. As $L \circ (H \circ G) = (L \circ H) \circ G$ is a unique precluster, then the two clusters containing it are equal.

With the notation in Remark 2.3, the embedding $I_{\mathscr{C}}$ is given by

$$I = I_{\mathscr{C}} : \mathscr{C} \to \mathrm{Clu}(\mathscr{C}) : (f : C \longrightarrow C') \mapsto I(f) = \left(u_f^{\mathscr{C}} : u_C^{\mathscr{C}} \longrightarrow u_{C'}^{\mathscr{C}}\right)$$

For the fourth point, we have to prove $(H^\sigma \circ G^\sigma)^\sigma \subset (H \circ G)^\sigma$, as the other inclusion is clear. So we consider $h \in H^\sigma$ and $g \in G^\sigma$, and $z \in [hg]_p$, with $h \in [h_0]_p$, $g \in [g_0]_p$, with $h_0 \in H, g_0 \in G$. If the target of g_0 is a q_0, then, given a $h_0 \in H$ with source this q_0, by the "parallelization lemma" (Lemma 3.7) with a chosen element h_{q_0} with source q_0, we get a zigzag from h_0 to h_{q_0}, with respect to the connection between g and g_0. This implies that $z \in [h_{q_0} g_0]_p$ i.e. it is an element of $(H \circ G)^\sigma$. ∎

A corollary of Theorem 3.9, easy but useful in application, is the following:

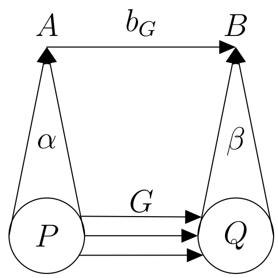
3.10. LEMMA. [Binding for action of preclusters on cocones] *Let* $G : P \longrightarrow Q$ *be a precluster,* G^σ *the cluster generated by* G, $\alpha = (\alpha_p : P(p) \longrightarrow A)_{p \in \mathrm{Ob}(\mathscr{P})}$ *a colimit cocone, and* $\beta = (\beta_p : Q(q) \longrightarrow B)_{q \in \mathrm{Ob}(\mathscr{Q})}$ *a cocone.*

With $\widehat{\alpha}$ *and* $\widehat{\beta}$ *the clusters associated to cocones* α *and* β *(Remark 3.3), with the compositions* $\circ$ *and* $\odot$ *of Proposition 3.8 and Theorem 3.9, we have*

$$\widehat{\beta} \odot G^\sigma = (\widehat{\beta} \circ G)^\sigma,$$

and there exists a unique "binding" morphism $b_G : A \longrightarrow B$, from the peak of α to the peak of β, such that $\widehat{\beta} \odot G^{\sigma} = b_G \odot \widehat{\alpha}$ (the cocone generated by $(b_G\alpha_p)_{p\in\mathscr{P}}$).

For any $h : A \to B$ and any $G : P \longrightarrow Q$, if $h = b_G$, that is to say if $\widehat{\beta} \odot G^{\sigma} = h \odot \widehat{\alpha}$, then h is determined by the family $(h\alpha_p)_{p\in\mathscr{P}}$ and for each p there is a $g : p \rightharpoonup q \in G$ such that: $h\alpha_p = \beta_q g$.



3.11. PROPOSITION. [Lim-Colim formula] *Given two small diagrams P and Q to $\mathscr{C}$ and the set $\mathrm{Clu}\,(\mathscr{C})(P,Q)$ of clusters from P to Q, we have a bijection*

$$\mathrm{Clu}\,(\mathscr{C})(P,Q) \simeq \underset{p\in\mathscr{P}}{\mathrm{Lim}}\underset{q\in\mathscr{Q}}{\mathrm{Colim}}\mathrm{Hom}_{\mathscr{C}}\,(P(p), Q(q))\,.$$

PROOF. If X is an object of $\mathscr{C}$, and $Q : \mathscr{Q} \longrightarrow \mathscr{C}$ a functor, then we have

$$\underset{q\in\mathscr{Q}}{\mathrm{Colim}}\mathrm{Hom}_{\mathscr{C}}\,(X, Q(q)) \simeq \biguplus_{q\in\mathscr{Q}} \mathrm{Hom}_{\mathscr{C}}\,(X, Q(q))\,/\equiv,$$

where $\equiv$ is the "zigzag relation", that is, the equivalence relation generated by $x \equiv xQ(q)$ for any $x : X \to Q(b)$ and $b : q \longrightarrow q'$. Therefore,

$$\underset{q\in\mathscr{Q}}{\mathrm{Colim}}\mathrm{Hom}_{\mathscr{C}}\,(X, Q(q)) \simeq \pi_0\,((X \downarrow Q))\,.$$

Hence, for any p in $\mathscr{P}$, we obtain

$$\underset{q\in\mathscr{Q}}{\mathrm{Colim}}\mathrm{Hom}_{\mathscr{C}}\,(P(p), Q(q)) \simeq \pi_0\left(\left(P_{|p} \downarrow Q)\right)\right) = \mathrm{CC}_{P,Q}(p),$$

$$\underset{p\in\mathscr{P}}{\mathrm{Lim}}\underset{q\in\mathscr{Q}}{\mathrm{Colim}}\mathrm{Hom}_{\mathscr{C}}\,(P(p), Q(q)) \simeq \underset{p\in\mathscr{P}}{\mathrm{Lim}}\mathrm{CC}_{P,Q}(p).$$

An element in this Lim-Colim is a $([g]_p)_{p\in\mathscr{P}} \in \prod_{p\in\mathscr{P}} \mathrm{CC}_{P,Q}(p)$ such that for all $a : p \to p'$ we have $\mathrm{CC}\,(a)\,([g]_p) = [gP(a)]_{p'}$. According to Proposition 3.4 and Definition 3.1, this element is a cluster. ∎

For any locally small category $\mathscr{C}$, with the Yoneda functor $h_{\mathscr{C}} : \mathscr{C} \to \mathbf{Set}^{\mathscr{C}^{\mathrm{op}}} = \widehat{\mathscr{C}}$ given by $h_{\mathscr{C}}(W) = \mathrm{Hom}_{\mathscr{C}}(-, W)$, Grothendieck in [Grothendieck, 1959] associates to any small diagram $P : \mathscr{P} \to \mathscr{C}$, the presheaf $L_{\mathscr{C}}(P) : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ given by

$$L_{\mathscr{C}}(P) = \mathrm{Colim}(h_{\mathscr{C}} \circ P),\;\; L_{\mathscr{C}}(P)(Z) = \mathrm{Colim}_p\mathrm{Hom}_{\mathscr{C}}(Z, P(p)) = \pi_0\,((Z \downarrow P))\,.$$

3.12. PROPOSITION. [Natural transformation $L_{\mathscr{C}}(G)$ associated to a cluster G] *Let P and Q be two small diagrams to $\mathscr{C}$. To any cluster $G : P \to Q$, there is an associated natural transformation $L_{\mathscr{C}}(G) : L_{\mathscr{C}}(P) \Rightarrow L_{\mathscr{C}}(Q)$ such that*

$$L_{\mathscr{C}} : \mathrm{Clu}\,(\mathscr{C})(P, Q) \simeq \mathrm{Hom}_{\widehat{\mathscr{C}}}\big(L_{\mathscr{C}}(P), L_{\mathscr{C}}(Q)\big)$$

is a bijection whose reciprocal is denoted by clu.

PROOF. For any cluster $G : P \to Q$, and with any choice $\Gamma = \{g_p \mid p \in \mathscr{P}\}$ of arrows $g_p \in G(p)$ as in Lemma 3.6, Item 4 — with therefore $G = \Gamma^\sigma$, we define a natural transformation $\theta_\Gamma : L_{\mathscr{C}}(P) \Rightarrow L_{\mathscr{C}}(Q)$ whose components are:

$$\theta_\Gamma(Z) : \left\{ \begin{array}{ccc} L_{\mathscr{C}}(P)(Z) & \longrightarrow & L_{\mathscr{C}}(Q)(Z) \\ [u]_{(Z \downarrow P)} & \longmapsto & [g_p u]_{(Z \downarrow Q)} \end{array} \right.$$

So Γ is seen as a presentation of θ_Γ, and in fact θ_Γ depends only on G, in such a way that it could be denoted as $\theta_\Gamma = L_{\mathscr{C}}(G)$, and then the cluster G can be seen as the "full or regular presentation" of θ_Γ, and denoted by $G = \mathrm{clu}(\theta_\Gamma)$.

Conversely, given an arbitrary natural transformation $\lambda : L_{\mathscr{C}}(P) \to L_{\mathscr{C}}(Q)$, there is a unique cluster $\mathrm{clu}(\lambda)$ such that $\lambda = L_{\mathscr{C}}(\mathrm{clu}(\lambda))$, determined by elements ξ_p in $\mathrm{Colim}_q \mathrm{Hom}(P(p), Q(q))$ (see Proposition 3.11), namely $\xi_p = \theta_{-,p}$, with $\theta_{X,p}$ the restriction of θ_X to $\mathrm{Hom}(X, P(p))$. ∎

3.13. PROPOSITION. *With Proposition 3.12 where $L_{\mathscr{C}}$ and* clu *are introduced, the elementary composition given in Theorem 3.9 by $H \odot G = (H \circ G)^\sigma$ can be recovered as*

$$H \odot G = \mathrm{clu}\big(L_{\mathscr{C}}(H) \circ L_{\mathscr{C}}(G)\big).$$

Hence Clu $(\mathscr{C})$ *is isomorphic to the category* LClu $(\mathscr{C})$ *whose objects are small diagrams P (and* <u>*not*</u> *the corresponding $L_{\mathscr{C}}(P)$), but where morphisms from P to Q are natural transformations from $L_{\mathscr{C}}(P)$ to $L_{\mathscr{C}}(Q)$. The category* LClu $(\mathscr{C})$ *is only equivalent to $\widehat{\mathscr{C}}_{\mathrm{small}}$, the full category of presheaves which are colimits of small diagrams. We have the factorisation*

$$L_{\mathscr{C}} : \mathrm{Clu}\,(\mathscr{C}) \simeq \mathrm{LClu}\,(\mathscr{C}) \longrightarrow \widehat{\mathscr{C}}_{\mathrm{small}} = \big(\mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}\big)_{\mathrm{small}} \to \mathbf{Set}^{\mathscr{C}^{\mathrm{op}}} = \widehat{\mathscr{C}}$$

4. The universal property of Clu $(\mathscr{C})$

4.1. DEFINITION. [Strict Free Cocompletion] *Given a locally small category $\mathscr{C}$, a* strict free cocompletion *of $\mathscr{C}$ is a universal datum $(I, \mathscr{F}, \lambda)$ where $\mathscr{F}$ is a locally small category which is cocomplete (any small diagram in it admits a colimit cocone), $I : \mathscr{C} \to \mathscr{F}$ is a functor such that, for every small diagram $P : \mathscr{P} \to \mathscr{C}$, the diagram $I \circ P = IP$ has a colimit $\lambda(IP)$ with colimit cocone $\lambda^P : IP \Rightarrow \lambda(IP)$ in $\mathscr{F}$. Hence, for every similar triple $(J, \mathscr{G}, \mu)$, where $J : \mathscr{C} \to \mathscr{G}$ is a functor toward a cocomplete category $\mathscr{G}$, and such that for every small diagram $P : \mathscr{P} \to \mathscr{C}$, the diagram JP admits a colimit $\mu(JP)$ with colimit cocone $\mu^P : JP \Rightarrow \mu(JP)$ in $\mathscr{G}$, there is a unique functor $\overline{J} : \mathscr{F} \to \mathscr{G}$ such that we have:*

$$\overline{J} \circ I = J \text{ and } \overline{J}(\lambda^P) = \mu^P.$$

If such a universal datum exists, the category $\mathscr{F}$ is unique up to an isomorphism.

4.2. REMARK. [Strict or Loose Free Cocompletion ?] The property invoked in Definition 4.1 is said *strict*. Another problem (said vague or loose) would be to only impose $\overline{J}$ to be unique up to an isomorphism of functors. A solution for the strict problem is a category determined up to an isomorphism, and it is also a solution for the loose problem, but any solution for this loose problem is a category determined only up to an equivalence. In both cases, the term "*free*" means that the functor I is free to preserve or not certain colimits or limits that may exist and/or be specified in $\mathscr{C}$.

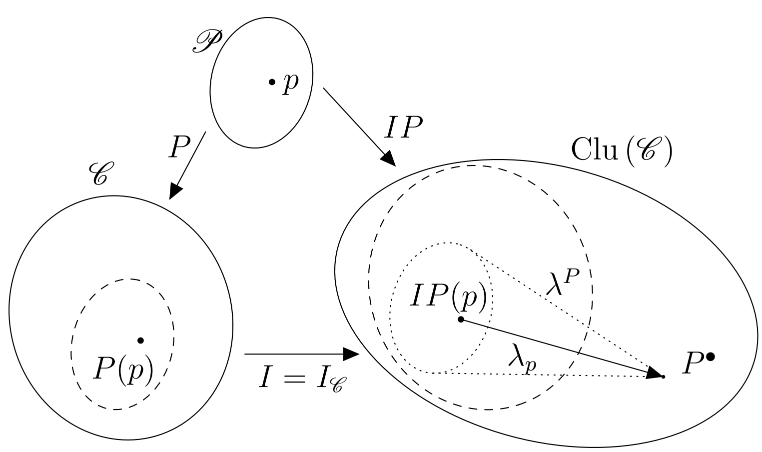
4.3. NOTATION.

1. *We will use the embedding*
$$I = I_{\mathscr{C}} : \mathscr{C} \to \mathrm{Clu}\,(\mathscr{C})$$
as described in Remark 2.3 and Item 2 of Theorem 3.9.
Thus, with $P : \mathscr{P} \to \mathscr{C}$, and $IP = I \circ P$, we have a diagram in $\mathrm{Clu}\,(\mathscr{C})$*:*
$$IP : \mathscr{P} \to \mathrm{Clu}\,(\mathscr{C}) : p \mapsto IP(p) : \{P(p)\} \to \mathscr{C} : P(p) \mapsto P(p).$$

2. *When we consider a small diagram $P : \mathscr{P} \to \mathscr{C}$ as an object of* $\mathrm{Clu}\,(\mathscr{C})$*, it will be denoted by $P^\bullet$ (see diagram below) unless no confusion is possible.*

3. *Of course, we must not confuse IP and $P^\bullet$. We define the cocone*
$$\lambda^P_{\mathscr{C}} = \lambda^P : IP \Rightarrow P^\bullet$$
by setting
$$\lambda^P = (\lambda_p)_{p \in \mathrm{Ob}(\mathscr{P})}, \quad \lambda_p : IP(p) \to P^\bullet, \quad \lambda_p(P(p)) = [\mathrm{id}_{P(p)}]_{P(p)}$$
Hence, λ_p is the cluster with $\lambda_p(P(p))$ being the connected component of $\mathrm{id}_{P(p)}$ *in $(IP(p) \downarrow P)$.*

The following figure summarizes the notations above:



4.4. THEOREM. *(ESFC) [A. Ehresmann, 1981] With Notation 4.3, for any given locally small category $\mathscr{C}$, we can explicitly construct, in an "elementary" way, a strict free cocompletion (cf. Definition 4.1) denoted by*

$$(I_{\mathscr{C}}, \mathrm{Clu}\,(\mathscr{C})\,, \lambda_{\mathscr{C}})$$

and the category of clusters $\mathrm{Clu}\,(\mathscr{C})$ *is called the* Elementary Strict Free Cocompletion *of* $\mathscr{C}$.

Hence, for any small diagram $P : \mathscr{P} \to \mathscr{C}$ we have

$$\overline{J}(P^{\bullet}) = \mu(JP).$$

Moreover, for any arbitrary small diagram $R : \mathscr{R} \to \mathrm{Clu}\,(\mathscr{C})$, we have an isomorphism

$$\overline{J}(\mathrm{Colim}(R)) \simeq \mathrm{Colim}(\overline{J} \circ R).$$

This construction is an extension of the construction of $\mathrm{Ind}(\mathscr{C})$.

4.5. REMARK. In fact the theorem was stated in dual form in [A. Ehresmann, 1981, p. 370-371], with the notion of *atlas*, dual to the notion of *cluster*, and the category $\mathrm{Atlas}(\mathscr{C}) = \left(\mathrm{Clu}(\mathscr{C}^{\mathrm{op}})\right)^{\mathrm{op}}$. The notion named "clusters" and the associated theorem are then stated in [Ehresmann & Vanbremeersch, 1987]. Of course, the cocompletion properties of $\mathrm{Clu}\,(\mathscr{C})$ in Theorem 4.4 correspond to the completion properties of $\mathrm{Atlas}(\mathscr{C})$. A final point, in order to appreciate the issues at stake, is the "elementary" or non-elementary nature of the explicit solution provided: its elements (the clusters), their mode of generation, their composition, the category $\mathrm{Clu}\,(\mathscr{C})$ they form, and also the effective calculation of the colimits in this category.

We now prove Theorem 4.4, point by point, through propositions and corollaries 4.6 to 4.11.

4.6. PROPOSITION. [Colimits in $\mathrm{Clu}\,(\mathscr{C})$ for diagrams in $\mathscr{C}$] *Given $P : \mathscr{P} \to \mathscr{C}$ a diagram in $\mathscr{C}$, then diagram IP from $\mathscr{P}$ to $\mathrm{Clu}\,(\mathscr{C})$ admits the object $P^{\bullet}$ (that is to say P itself) of* $\mathrm{Clu}\,(\mathscr{C})$ *as colimit in* $\mathrm{Clu}\,(\mathscr{C})$, *with a colimit cocone*

$$\lambda^{P} : IP \to \lambda(IP) = P^{\bullet}.$$

PROOF. For $a : p' \to p \in \mathscr{P}$ we have a cluster $(IP)(a) : (IP)(p') \to (IP)(p)$ given by $(IP)(a)(P(p')) = [P(a)]_{P(p')}$, the connected component of $P(a)$ in $(IP(p') \downarrow IP(p))$. For each p we have the cluster $\lambda_p : IP(p) \to P^{\bullet}$. With the composition of clusters given in Corollary 3.9, these clusters verify, for every $a : p' \to p$:

$$\lambda_p \odot (IP)(a) = \lambda_{p'},$$

as we verify: $\left(\lambda_p \odot (IP)(a)\right)(P(p')) = [\mathrm{id}_{P(p)} P(a)]_{P(p')} = [P(a)]_{P(p')} = \lambda_{p'}(P(p'))$.
So $(\lambda_p)_{p \in \mathrm{Ob}(\mathscr{P})}$ determines a cocone

$$\lambda^{P} : IP \to P^{\bullet}.$$

Now we verify that this cocone is a colimit in $\mathrm{Clu}(\mathscr{C})$. Let μ be a cocone in $\mathrm{Clu}(\mathscr{C})$ from IP to $Q^\bullet$. We are looking for a cluster $G : P^\bullet \to Q^\bullet$ such that $G \odot \lambda^P = \mu$, i.e. such that $(G\odot\lambda_p)_p = \mu_p$. Let $g : P(p) \to Q(q)$. If $(g : p \rightharpoonup q) \in G(p)$, then $(G\odot\lambda_p)(P(p)) = \mu(P(p))$ is represented by $[g\,\mathrm{id}_{P(p)}]_{P(p)} = [g]_{P(p)}$, and then $g \in \mu_p(P(p))$. And conversely. So $G(p) = \mu_p(P(p))$ Since μ is a cocone, for any $a : p \to p'$, we have $\mu_{p'} \odot (IP)(a) = \mu_p$. Therefore, if $g' \in \mu_{p'}(P(p'))$ then $g'P(a)$ is in $\mu_p(P(p))$. This last statement is equivalent to saying that if $g' \in G(p')$ then $g'P(a)$ is in $G(p)$, i.e. $[g'P(a)]_p = [g]_p$. This proves that G is functorial in that it satisfies (CLU-3). ∎

4.7. PROPOSITION. [Unicity of $\overline{J}$] *With the same notation as in Proposition 4.6, given a functor $J : \mathscr{C} \to \mathscr{G}$ — as in Definition 4.1 — we have a unique $\overline{J}$ such that*

$$\overline{J} \circ I = J \text{ and } \overline{J}(\lambda^P) = \mu^P.$$

PROOF. The values of $\overline{J}$ on objects are:

$$\overline{J}(P^\bullet) = \overline{J}(\lambda(IP)) = \mu(JP).$$

If $G : P^\bullet \to Q^\bullet$ is a cluster over $\mathscr{C}$, let b_{JG} be the binding of the precluster JG in $\mathscr{G}$ (see Lemma 3.10). We have:

$$\overline{J}(G) = b_{JG} : \mu(JP) \to \mu(JQ).$$

We must prove that for every p we have $\overline{J}(G)\mu^P_p = b_{JG}\mu^P_p$. As G is a cluster, IG is a precluster from IP to IQ, and, with Proposition 4.6, $G = b_{IG}$, the binding of IG (see Lemma 3.10). Hence, G is determined by the family of the $G\lambda^P_p = b_{IG}\lambda^P_p$, and we know that there is a $g : p \rightharpoonup q$ such that $G\lambda^P_p = b_{IG}\lambda^P_p = \lambda^Q_q I(g)$. Then $\overline{J}(G)\mu^P_p = \overline{J}(G)\overline{J}(\lambda^P_p) = \overline{J}(G\lambda^P_p) = \overline{J}(\lambda^Q_q I(g)) = \overline{J}(\lambda^Q_q)\overline{J}(I(g)) = \mu^Q_q J(g) = b_{JG}\mu^P_p$. ∎

To complete the proof of Theorem 4.4, it remains to show Proposition 4.9 below and its corollaries. Proposition 4.9 gives a detailed proof of the procedure E_R indicated in [A. Ehresmann, 1981, p. 371] and that we now quote, in dual form and with modifications of her notations, in the next Proposition 4.8.

4.8. PROPOSITION. [Cocompletion by the procedure E_R] *For $R : \mathscr{R} \to \mathrm{Clu}(\mathscr{C})$ with $\mathscr{R}$ small, its colimit is $E^\bullet_R$, that is to say the functor $E_R : \mathscr{E}_R \to \mathscr{C}$ defined as follows: $\mathscr{E}_R$ contains as a subcategory the disjoint union of the categories $\mathscr{I}_r$ domain of $R(r)$, for each r, and its other morphisms from i in $\mathscr{I}_r$ to i' in $\mathscr{I}_{r'}$ are the g which belong to the cluster $R(\rho)$ for some $\rho : r \to r'$ in $\mathscr{R}$; the functor E_R extends the functors $R(r)$ and maps $g : i \to i'$ on ρ.*

In this procedure, $\mathscr{E}_R$ is a kind of lax-colimit, analogous to the construction of fibrations by Grothendieck or of Ehresmann's categories of hypermorphisms.

Given $P : \mathscr{P} \to \mathscr{C}$, we see that

$$\mathscr{E}_{IP} \simeq \mathscr{P}, E_{IP} \simeq P.$$

Therefore, the simple construction considered in Proposition 4.6 results from that introduced in Proposition 4.8.

In our proof of Theorem 4.4, and as announced above, we are going to demonstrate Proposition 4.9, which also establishes Proposition 4.8 by considering in detail the sufficient cases of sums and coequalizers.

4.9. PROPOSITION. [Cocompletion] *For any locally small category $\mathscr{C}$, the locally small category* $\mathrm{Clu}(\mathscr{C})$ *is cocomplete, i.e. admits small colimits.*

PROOF. With respect to the procedure E_R recalled in Proposition 4.8, we explicitly describe colimits, in the cases of sums and coequalizers (of which of course the general case follows).

1. Given a set K and a family of diagrams $P_k : \mathscr{P}_k \to \mathscr{C}$, set $\mathscr{S} = \coprod_{k \in K} \mathscr{P}_k$ with canonical injections $I_k : \mathscr{P}_k \to \mathscr{S}$. The sum in $\mathrm{Clu}(\mathscr{C})$ of the diagrams $P_k : \mathscr{P}_k \to \mathscr{C}$ is the diagram $S : \mathscr{S} \to \mathscr{C}$, whose canonical injections are the clusters $i_k = [I_k, \mathrm{Id}_{P_k}]$ generated by the morphisms of diagrams (I_k, Id_{P_k}) (as in Example 3.3) and given, for any $p \in \mathscr{P}_k$, by $i_k(p) = \left[\mathrm{Id}_{P_k(p)}\right]_p$.

2. Given two diagrams $P : \mathscr{P} \to \mathscr{C}$ and $Q : \mathscr{Q} \to \mathscr{C}$, and two clusters $A, B : P \to Q$, the coequalizer of A and B is a cluster $Z : Q \to W$ given as follows. W is a diagram $W : \mathscr{W} \to \mathscr{C}$, where $\mathscr{W}$ is the category whose set of objects is the disjoint union $\mathrm{Ob}(\mathscr{P}) \coprod \mathrm{Ob}(\mathscr{Q})$ and whose generating morphisms are the copies of morphisms of $\mathscr{P}$, $\mathscr{Q}$ and the elements of $A \cup B$, a morphism from p to q being a $w : p \rightharpoonup q$ in A or in B. The functor W extends P and Q, and maps $w : p \rightharpoonup q$ in A or in B onto itself. Then, as above in the case of sums, Z is the cluster generated by the inclusion $\mathscr{Q} \hookrightarrow \mathscr{W}$, with values $Z(q) = [\mathrm{id}_{Q(q)}]_q$. If $H : Q \to R$ satisfies $H \odot A = H \odot B$, then for any p, with chosen elements $a : p \rightharpoonup q_a$, $h_a \in H(q_a)$, $b : p \rightharpoonup q_b$, $h_b \in H(q_b)$, we have $(H \odot A)(p) = [h_a a]_p = [h_b b]_p = (H \odot B)(p)$. Thus $h_a a$ and $h_b b$ are connected in $\left(P_{|p} \downarrow R\right)$. This means that in $\left(W_{|p} \downarrow R\right)$ the same data are connected. Hence H admits a unique factorization H' by Z, given by $H'(q) = [h]_q$ with $h \in H(q)$, and at the same time $H'(p) = [ha]_p$ with $a : p \rightharpoonup q \in A(p)$ and thus $a \in \mathscr{W}$, and also $H'(p) = [hb]_p$ with $b : p \rightharpoonup q \in B(p)$ and thus $b \in \mathscr{W}$. ∎

4.10. COROLLARY. *In Definition 4.1, for any small diagram $R : \mathscr{R} \to \mathscr{F}$ that is not necessarily of the form $R = IP$, we nevertheless have an isomorphism*

$$\overline{J}(\mathrm{Colim}(R)) \simeq \mathrm{Colim}(\overline{J} \circ R).$$

PROOF. As $\mathscr{F}$ is determined up to an isomorphism, it is enough to prove the corollary in the case of $\mathrm{Clu}(\mathscr{C})$, with $R : \mathscr{R} \to \mathrm{Clu}(\mathscr{C})$. So let $E_R : \mathscr{E}_R \longrightarrow \mathscr{C}$ be its colimit in $\mathrm{Clu}(\mathscr{C})$. We have $\mathrm{Colim}_{\mathscr{R}}(R) = E_R^\bullet \simeq \mathrm{Colim}_{\mathscr{E}_R}(IE_R)$, and so

$$\overline{J}(\mathrm{Colim}_{\mathscr{R}}(R)) = \overline{J}(E_R^\bullet) \simeq \overline{J}(\mathrm{Colim}_{\mathscr{E}_R}(IE_R)) \simeq \overline{J}(\lambda(IE_R)) = \mu(JE_R) \simeq \mathrm{Colim}_{\mathscr{E}_R}(JE_R).$$

Also we have for any object r in $\mathscr{R}$ (and consequently in $\mathscr{E}_R$):

$$(\overline{J}R)(r) = \overline{J}(R(r)) \simeq \overline{J}(\lambda(IR(r)) = \mu(JR(r)) \simeq \mathrm{Colim}_{\mathscr{I}_r}(JR(r)).$$

In fact the source $\mathscr{E}_R$ of the colimit E_R, as described in Proposition 4.8, is a kind of lax-colimit, and that allows to have

$$\mathrm{Colim}_{\mathscr{E}_R}(JE_R) \simeq \mathrm{Colim}_{\mathscr{R}}\left(\mathrm{Colim}_{\mathscr{I}_r} JR(r)\right).$$

■

4.11. COROLLARY. *The category* $\mathrm{Ind}(\mathscr{C})$ *is a full subcategory of* $\mathrm{Clu}\,(\mathscr{C})$ *— or of* $\mathrm{LClu}\,(\mathscr{C})$*. — and the cocompletion result of Grothendieck is a particular case of Theorem 4.4.*

PROOF. The corollary follows from two remarks. Firstly, computations of colimits in Proposition 4.9 depend only on the general process E_R (see Proposition 4.8), from which it is clear that filtered colimits of filtered diagrams are filtered diagrams. Secondly, rather than working with $\mathrm{Clu}\,(\mathscr{C})$ it is possible to use $\mathrm{LClu}\,(\mathscr{C})$. ■

5. Conclusion: elements for comparison with other cocompletions

As the construction developed here offers some details of a proof of Andrée Ehresmann's theorem given in 1981, we must compare her initial statement with what was done at that time and before. The reader will find more references in the bibliography in [A. Ehresmann, 1981], among which we will mention only a few, in chronological order: [Grothendieck, 1959], [Isbell, 1960], [Grothendieck & Verdier, (1961) 1972], [Tsalenko, 1963], [Roux, 1964], [Lambek, 1966], [Duskin, 1966], [Trnkova, 1966], [C. Ehresmann, 1967], [Isbell, 1967], [Freyd, 1969], [Gabriel & Ulmer, 1971], [Conduché, 1971], [Prochasson, 1972], [Bastiani & Ehresmann, 1972], [Borceux, 1972], [Deleanu & Hilton, 1976], [Porter, 1979], [Adamek, Herrlich & Strecker, 1979], [Street, 1979], [Adamek & Koubek, 1981], [Kelly, 1982], [Johnstone & Joyal, 1982].

If someone wanted to write a complete history of cocompletions, they would have to take into consideration all these references, together with those given in [A. Ehresmann, 1981], as well as others in the history of lattice theory, and also of course, additional works such as [Velebil & Adámek, 2002], [Day & Lack, 2007], or more recently [P. Perrone & W. Tholen, 2021]. But this is not our goal here; we only want to extract from the aforementioned references some salient points to appreciate the construction of clusters. In this respect, we do not tackle here the question of preserving limits (conservative or preservative cocompletions), which occurs in almost all the studies referred above. We are only interested in the very nature of cocompletions, their effectiveness, and the precise determination of the problem they solve.

Usually, it is considered that the "best" description of a free cocompletion of a category $\mathscr{C}$ is the category of small presheaves on $\mathscr{C}$, i.e. all small colimits of hom-functors, and especially the free cocompletion under filtered colimits then consists of small filtered colimits of hom-functors. Indeed, after Yoneda's lemma, following [Grothendieck, 1959], most of the works of the 1960s seem to go in this direction, e.g. [Isbell, 1960], [Lambek, 1966], etc.

This is certainly true if we presuppose two points: that we are interested in cocompletion only up to an equivalence of category (loose cocompletion), and if the question of the effectiveness of presentations and computations in the said cocompletion remains secondary (and, in fact, these two points are related).

But in fact the question is intricate, as we already see in the description of the free cocompletion under filtered colimits as $\mathrm{Ind}(\mathscr{C})$ in [Grothendieck, 1959]. We encounter the "natural" Lim-Colim formula, the difficulty of handling the elements, and especially performing the composition. For applications to Shape Theory, [Deleanu & Hilton, 1976] succeeded in simplifying this question, and this will be commented by Porter [Porter, 1979], and [Johnstone & Joyal, 1982], at least in the filtered case. Therefore, in this case, they reach a strict solution to the problem, although they do not seem to notice it since they explicitly identify their $\mathrm{Ind}(\mathscr{C})$ with the category of small filtered presheaves on $\mathscr{C}$. It is worth emphasizing that this identification is not an isomorphism but an equivalence.

And now, what about the general case of diagrams that are not necessarily filtered? The solution proposed by [A. Ehresmann, 1981] seems adequate. The decisive idea is therefore to replace the Deleanu-Hilton or Johnstone-Joyal description of the filtered case by the explicit consideration of zigzags and the associated connected components. This is not a mere obvious and even gratuitous generalization, but the actual way to proceed, as further detailed in the next paragraphs. Moreover, the idea even originates from other sources, namely from works on cocompletions in sketch theory [C. Ehresmann, 1966], [C. Ehresmann, 1967], [Bastiani & Ehresmann, 1972], and achievements of Charles Ehresmann on the categorical treatment of local structures and manifolds [C. Ehresmann, 1952, 1954], [C. Ehresmann, 1958], by introducing the notion of atlas [C. Ehresmann, 1964]. So the notion of cluster (or dually of atlas), as introduced in [A. Ehresmann, 1981], is a unification of two apparently disjoint ideas : the notion of ideal in a lattice, the notion of atlas for a manifold.

That said, the categories $\mathrm{Clu}\,(\mathscr{C})$ and $\widehat{\mathscr{C}}_{\text{small}} = \left(\mathbf{Set}^{\mathscr{C}^{\text{op}}}\right)_{\text{small}}$ are equivalent but not isomorphic, via a canonical comparison functor between the strict and loose solutions. It is, with the notation $L_{\mathscr{C}}$ from Proposition 3.12 and Proposition 3.13):

$$L_{\mathscr{C}} : \mathrm{Clu}\,(\mathscr{C}) \longrightarrow \widehat{\mathscr{C}}_{\text{small}}.$$

The reader can compare the ease of calculating colimits in $\mathrm{Clu}\,(\mathscr{C})$, for a functor $R : \mathscr{R} \to \mathrm{Clu}\,(\mathscr{C})$ (this colimit being $E_R : \mathscr{E}_R \to \mathscr{C}$ in Proposition 4.8 and Proposition 4.9, where $\mathscr{E}_R$ is simply a lax-colimit) with the fact that in the category $\left(\mathbf{Set}^{\mathscr{C}^{\text{op}}}\right)_{\text{small}}$ we have to resort to a model of Set Theory (the category $\mathbf{Set}$), and then we have to use quotients (to do the point by point colimits in $\mathbf{Set}$). The same holds for our variant $\mathrm{LClu}\,(\mathscr{C})$ (Proposition 3.13). In a sense, one can imagine that in $\mathrm{Clu}\,(\mathscr{C})$, the quotients modulo the zigzags are encapsulated in the prior fabrication of the morphisms.

Our presentation has emphasized that we can construct cocompletion via the initial data of π_0 only and that the computations are actually feasible with some elementary material and notation such as $[g]_p$ and G^σ, which are definable from π_0. Thus, the compositions of arrows and the calculations of colimits become very easy and effective. And this is crucial for concrete applications, in the vein of [Ehresmann & Vanbremeersch, 2007] or [Pastor, Beurier, Ehresmann, Waldeck, 2020].

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